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$N = 1$ Superconformal Symmetry in Four-dimensions

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Abstract

The $N = 1,\ d = 4$ superconformal group is studied and its representations are discussed. Under superconformal transformations, left invariant derivatives and some class of superfields, including *supercurrents*, are shown to follow these representations. In other words, these superfields are *quasi-primary* by analogy with two dimensional conformal field theory. Based on these results, we find the general forms of the two-point and the three-point correlation functions of the quasi-primary superfields in a group theoretical way. In particular, we show that the two-point function of the supercurrent is unique up to a constant and the general form of the three-point function of the supercurrent has two free parameters.

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1 Introduction

Four dimensional conformal field theories, CFT_4 are of special physical interest for the standard relativistic quantum field theories. Quantum field theories at renormalization group fixed points are expected to be conformally invariant. $N = 4$ super Yang-Mills theory in four dimensions was shown to be superconformally invariant since its β-function vanishes, and the β-function is proportional to the trace anomaly [1–6]. A few other four dimensional supersymmetric conformal field theories, SCFT_4 have been known as well - certain $N = 1$ supersymmetric theory [7], $N = 2$ $SU(2)$ QCD [8, 9] and some $N = 2$ models [10]. In particular a number of studies on SCFT_4 [11, 12] have been done with regard to the electro-magnetic duality [13] where the duality is realized as infrared fixed points of ordinary supersymmetric theories. Further work has been done on correlation functions of superfields [14–21][1] and operator product expansions in SCFT_4 [23–25] based on the superconformal Ward identities, where the infinitesimal superconformal transformation rules for superfields play a crucial role. In this paper, we approach the problem in a group theoretical way, investigating not just infinitesimal superconformal symmetry generators but finite transformations. This method enables us to deal with non-scalar superfields in a compact way and so turns out to be useful to prove the uniqueness of the correlation functions and get results in closed forms. Our method can be found in some works on ordinary conformal field theory [26], but has not, to our knowledge, been applied in superconformal field theory. The organization of this paper is the following: In section 2, starting from the definition of superconformal group, we study the sufficient and necessary conditions for a supercoordinate transformation to be superconformal. We then derive all the superconformal group generators from an infinitesimal version of these conditions, namely the fundamental elements of superconformal group. They are known as supertranslations, superdilations, super Lorentz transformations and superinversion. We find 4×4, 2×2 matrix representations of superconformal group and show that they give the transformation rules for the left invariant derivatives and for a certain class of superfields, in particular the chiral/anti-chiral superfields and the supercurrents in Wess-Zumino model and also in vector superfield theory. In other words, these superfields are *quasi-primary* by analogy with two dimensional conformal field theory. In section 3, based on these results, we study the general forms of correlation functions of superfields, which follow the representations above under superconformal transformations and so give the superconformal invariance property to the correlation functions. In particular, we show that the two-point function of the supercurrent is unique up to a constant and the general form of the three-point function of the supercurrent has two free

[1] I am informed by Prof. P. West that an analysis of a superconformal Killing equations was also undertaken by B. Conlong [21] and that this will be discussed further in [22].

parameters. The work presented here is an extension of some results in ordinary conformal field theory [27, 28] to superconformal field theory. Throughout the paper we assume the real Minkowski spacetime, and so the real $N=1$ superconformal group, $SU(2,2|1)$ rather than the complex superconformal group, $SL(4|1;\mathbb{C})$. The latter case was discussed in detail in [29]. Nevertheless such a distinction is not relevant to our main result.

2 Superconformal Symmetry in Four-dimensions

2.1 Superconformal Group

Following the notations of Wess & Bagger [30], the $N=1$ supersymmetry algebra is

$$
\begin{gathered}
[P_\mu \ , \ P_\nu] = [P_\mu \ , \ Q_\alpha] = [P_\mu \ \ \overline{Q}_{\dot\alpha}] = \{Q_\alpha \ , \ Q_\beta\} = \{\overline{Q}_{\dot\alpha} \ , \ \overline{Q}_{\dot\beta}\} = 0 \\
\{Q_\alpha \ , \ \overline{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu
\end{gathered}
\tag{2.1}
$$

The element of super group is given as

$$
g(z) = e^{i(-x^\mu P_\mu + \theta^\alpha Q_\alpha + \overline{Q}_{\dot\alpha}\bar\theta^{\dot\alpha})} \tag{2.2}
$$

where $z^A = x^\mu, \theta^\alpha, \bar\theta^{\dot\alpha}$ are supercoordinates with the restriction

$$
\theta^\alpha = (\bar\theta^{\dot\alpha})^\dagger \tag{2.3}
$$

α and $\dot\alpha$ indices are raised or lowered by the antisymmetric 2×2 matrix ϵ, $\epsilon_{12} = \epsilon^{21} = 1$, thus $Q^\alpha = \epsilon^{\alpha\beta} Q_\beta$ etc. The supersymmetric interval, x_{12} between z_1 and z_2 is defined by $g(z_{12}) = g^{-1}(z_2) g(z_1)$ and so has the form

$$
\begin{aligned}
&x_{12} = x_1 - x_2 - i\theta_1\sigma\bar\theta_2 + i\theta_2\sigma\bar\theta_1 \\
&\theta_{12} = \theta_1 - \theta_2 \qquad \bar\theta_{12} = \bar\theta_1 - \bar\theta_2
\end{aligned}
\tag{2.4}
$$

Left invariant derivatives $D_\alpha, \overline{D}_{\dot\alpha}$ are

$$
D_\alpha = \partial_\alpha + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha} = \partial^-_\alpha \qquad \overline{D}_{\dot\alpha} = -\bar\partial_{\dot\alpha} - i\theta^\alpha\partial_{\alpha\dot\alpha} = -\bar\partial^+_{\dot\alpha} \tag{2.5}
$$

where $\partial_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha}\partial_\mu$ and the superscripts, $\pm$, in the partial differential operators mean that $x_\pm = x \pm i\theta\sigma\bar\theta$ are taken to be fixed.
Infinitesimal supersymmetric interval, w is defined from eq.(2.4) as z_1 goes to z_2

$$
\mathsf{w} = dx + i\theta\sigma d\bar\theta - id\theta\sigma\bar\theta \tag{2.6}
$$

Superconformal group is the subgroup of supercoordinate transformations, $g: \ z \rightarrow z'$ that preserve the infinitesimal supersymmetric interval length, w^2 up to a local scale factor

$$g: \ z \rightarrow z' \quad \Rightarrow \quad \mathsf{w}^2 \ \rightarrow \ \mathsf{w}'^2 = \Omega^2(g;z)\mathsf{w}^2 \tag{2.7}$$

where $\Omega(g;z)$ is a local scale factor.
We consider continuous superconformal transformations as superconformal transformations which satisfy

$$\det\left(\tfrac{\partial^- \theta'}{\partial \theta}\right) \neq 0 \qquad\qquad \det\left(\tfrac{\partial^+ \bar{\theta}'}{\partial \theta}\right) \neq 0 \tag{2.8}$$

By the restriction $\theta'^\alpha = (\bar{\theta}'^{\dot{\alpha}})^\dagger$, one of these two inequalities implies the other. Later we will see that this condition is preserved under the successive continuous superconformal transformations so that they form a group, namely continuous superconformal group. - It will be shown that $\left(\frac{\partial^- \theta'}{\partial \theta}\right)$ is a representation of the continuous superconformal group.
Now we are at the position to state that for a supercoordinate transformation, $g: z \rightarrow z'$ which satisfies eq.(2.8), the necessary and sufficient conditions for g to be a (continuous) superconformal transformation are[2]

$$\begin{aligned} &x'_+(x_+,\theta), \quad \theta'(x_+,\theta) \qquad \text{functions of } x_+ \text{ and } \theta \text{ only} \\ &x'_-(x_-,\bar{\theta}), \quad \bar{\theta}'(x_-,\bar{\theta}) \qquad \text{functions of } x_- \text{ and } \bar{\theta} \text{ only} \end{aligned} \tag{2.9}$$

with the reality condition

$$x'_+(x_+,\theta) - x'_-(x_-,\bar{\theta}) = 2i\theta'(x_+,\theta)\sigma\bar{\theta}'(x_-,\bar{\theta}) \tag{2.10}$$

This reality condition should be satisfied by the definition of $x'_\pm$.
proof
Generally under supercoordinate transformation, $g: z \rightarrow z'$, w^μ transforms to

$$\mathsf{w}^\mu \rightarrow \mathsf{w}'^\mu = A^\mu{}_\nu(g;z)\mathsf{w}^\nu + B^{\mu\alpha}(g;z)d\theta_\alpha + \bar{B}^\mu_{\dot{\alpha}}(g;z)d\bar{\theta}^{\dot{\alpha}} \tag{2.11}$$

where

$$A^\mu{}_\nu(g;z) = \frac{\partial x'^\mu_+}{\partial x^\nu} - 2i\frac{\partial \theta'}{\partial x^\nu}\sigma^\mu\bar{\theta}' = \frac{\partial x'^\mu_-}{\partial x^\nu} + 2i\theta'\sigma^\mu\frac{\partial \bar{\theta}'}{\partial x^\nu} \tag{2.12}$$

[2]In section 2.9 of the book by I. Buchbinder & S. Kuzenko [31], they take these conditions as the definition of $N=1$ superconformal transformation and then show that the infinitesimal supersymmetric interval length, w^2 is preserved under such a transformation.

$$B^\mu_\alpha(g;z) = \partial^-_\alpha x'^\mu_- - 2i\theta'\sigma^\mu\partial^-_\alpha\bar\theta'$$
$$\bar B^\mu_{\dot\alpha}(g;z) = -\bar\partial^+_{\dot\alpha}x'^\mu_+ + 2i\bar\partial^+_{\dot\alpha}\theta'\sigma^\mu\bar\theta' = (B^\mu_\alpha(g;z))^\dagger \tag{2.13}$$

From the definition of superconformal group we require

$$A^\lambda{}_\mu\eta_{\lambda\rho}A^\rho{}_\nu \propto \eta_{\mu\nu}$$
$$B^\mu_\alpha B_{\mu\beta} = \bar B^\mu_{\dot\alpha}\bar B_{\mu\dot\beta} = A^\mu{}_\nu B_{\mu\alpha} = A^\mu{}_\nu\bar B_{\mu\dot\alpha} = B^\mu_\alpha\bar B_{\mu\dot\alpha} = 0 \tag{2.14}$$

Simple application of chain rule leads to

$$\frac{\partial x^\nu_+}{\partial x'^\lambda_+}A^\mu{}_\nu + \frac{\partial\theta^\alpha}{\partial x'^\lambda_+}B^\mu_\alpha + \frac{\partial\bar\theta^{\dot\alpha}}{\partial x'^\lambda_+}\bar B^\mu_{\dot\alpha} = \delta^\mu{}_\lambda \tag{2.15}$$

This makes eq.(2.14) simple. With matrix notation

$$A^t(g;z)\eta A(g;z) = \Omega^2(g;z)\eta$$
$$B^\mu_\alpha = \bar B^\mu_{\dot\alpha} = 0 \tag{2.16}$$

From $\bar\partial^+_{\dot\alpha}\bar B^\mu_{\dot\beta} + \bar\partial^+_{\dot\beta}\bar B^\mu_{\dot\alpha} = 0$ we get

$$\bar\partial^+_{\dot\alpha}\theta'^\gamma\bar\partial^+_{\dot\beta}\bar\theta'^{\dot\gamma} + \bar\partial^+_{\dot\beta}\theta'^\gamma\bar\partial^+_{\dot\alpha}\bar\theta'^{\dot\gamma} = 0 \tag{2.17}$$

By the assumption, $\det\left(\frac{\partial^+\bar\theta'}{\partial\theta}\right) \neq 0$, we can multiply $\left(\frac{\partial^+\bar\theta'}{\partial\theta}\right)^{-1}$ to eq.(2.17) to get

$$\bar\partial^+_{\dot\alpha}\theta'^\gamma\delta^{\dot\gamma}_{\dot\beta} + \bar\partial^+_{\dot\beta}\theta'^\gamma\delta^{\dot\gamma}_{\dot\alpha} = 0 \tag{2.18}$$

Putting $\dot\alpha = \dot\beta = \dot\gamma$ gives

$$\bar\partial^+_{\dot\alpha}\theta'^\beta = 0 \tag{2.19}$$

This and $\bar B^\mu_{\dot\alpha} = 0$ imply that x'_+ and θ' are functions of x_+ and θ only. Similarly one can show that x'_- and $\bar\theta'$ are functions of x_- and $\bar\theta$ only. Thus these are necessary conditions for g to be superconformal. Now we need to show that these actually imply $A^t(g;z)\eta A(g;z) \propto \eta$. Acting $\bar\partial^+_{\dot\alpha}\partial^-_\alpha$ to the reality condition (2.10) gives

$$\partial_{\alpha\dot\alpha}x'_+ - 2i\partial_{\alpha\dot\alpha}\theta'\sigma\bar\theta' = \partial^-_\alpha\theta'\sigma\bar\partial^+_{\dot\alpha}\bar\theta' \tag{2.20}$$

and so

$$A^{\mu}{}_{\nu} = -\tfrac{1}{2}\tilde{\sigma}^{\dot{\alpha}\alpha}_{\nu}\partial^{-}_{\alpha}\theta'\sigma^{\mu}\bar{\partial}^{+}_{\dot{\alpha}}\bar{\theta}' \tag{2.21}$$

$$A^{\lambda}{}_{\mu}\eta_{\lambda\rho}A^{\rho}{}_{\nu} = \tfrac{1}{2}\tilde{\sigma}^{\dot{\alpha}\alpha}_{\mu}\tilde{\sigma}^{\dot{\beta}\beta}_{\nu}\partial^{-}_{\alpha}\theta'^{\gamma}\partial^{-}_{\beta}\theta'_{\gamma}\bar{\partial}^{+}_{\dot{\alpha}}\bar{\theta}'_{\dot{\gamma}}\bar{\partial}^{+}_{\dot{\beta}}\bar{\theta}'^{\dot{\gamma}} \tag{2.22}$$

where $\tilde{\sigma}^{\mu\dot{\alpha}\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\sigma^{\mu}_{\beta\dot{\beta}}$. However from $\partial^{-}_{\alpha}\theta'^{\gamma}\partial^{-}_{\beta}\theta'_{\gamma} + \partial^{-}_{\beta}\theta'^{\gamma}\partial^{-}_{\alpha}\theta'_{\gamma} = 0$ we get

$$\partial^{-}_{\alpha}\theta'^{\gamma}\partial^{-}_{\beta}\theta'_{\gamma} = \tfrac{1}{2}\epsilon_{\alpha\beta}\partial^{-\delta}\theta'^{\gamma}\partial^{-}_{\delta}\theta'_{\gamma} \qquad \bar{\partial}^{+}_{\dot{\alpha}}\bar{\theta}'_{\dot{\gamma}}\bar{\partial}^{+}_{\dot{\beta}}\bar{\theta}'^{\dot{\gamma}} = \tfrac{1}{2}\epsilon_{\dot{\beta}\dot{\alpha}}\bar{\partial}^{+}_{\dot{\delta}}\bar{\theta}'_{\dot{\gamma}}\bar{\partial}^{+\dot{\delta}}\bar{\theta}'^{\dot{\gamma}} \tag{2.23}$$

Using $\sigma^{\mu}_{\alpha\dot{\alpha}}\tilde{\sigma}^{\dot{\alpha}\alpha}_{\nu} = -2\delta^{\mu}{}_{\nu}$, we finally have

$$A^{\lambda}{}_{\mu}\eta_{\lambda\rho}A^{\rho}{}_{\nu} = \tfrac{1}{4}(\partial^{-\delta}\theta'^{\gamma}\partial^{-}_{\delta}\theta'_{\gamma}\bar{\partial}^{+}_{\dot{\delta}}\bar{\theta}'_{\dot{\gamma}}\bar{\partial}^{+\dot{\delta}}\bar{\theta}'^{\dot{\gamma}})\eta_{\mu\nu} \propto \eta_{\mu\nu} \tag{2.24}$$

This completes our proof.
There is another type of superconformal transformation, which satisfies

$$\det\left(\frac{\partial^{-}\bar{\theta}'}{\partial\theta}\right) \neq 0 \qquad\qquad \det\left(\frac{\partial^{+}\theta'}{\partial\bar{\theta}}\right) \neq 0 \tag{2.25}$$

The necessary and sufficient conditions for such a supercoordinate transformation, g to be a superconformal transformation are

$$\begin{aligned} &x'_{+}(x_{-},\bar{\theta}), \quad \theta'(x_{-},\bar{\theta}) \qquad \text{functions of } x_{-} \text{ and } \bar{\theta} \text{ only} \\ &x'_{-}(x_{+},\theta), \quad \bar{\theta}'(x_{+},\theta) \qquad \text{functions of } x_{+} \text{ and } \theta \text{ only} \end{aligned} \tag{2.26}$$

with the reality condition

$$x'_{+}(x_{-},\bar{\theta}) - x'_{-}(x_{+},\theta) = 2i\theta'(x_{-},\bar{\theta})\sigma\bar{\theta}'(x_{+},\theta) \tag{2.27}$$

We will call this type of superconformal transformation "superinversion-type transformation".
Infinitesimal, therefore continuous, superconformal transformation, $g : z \to z' \simeq z + \delta z$ satisfies the infinitesimal reality condition

$$h(x,\theta,\bar{\theta}) = v(x_{+},\theta) - 2i\lambda(x_{+},\theta)\sigma\bar{\theta} = \bar{v}(x_{-},\bar{\theta}) + 2i\theta\sigma\bar{\lambda}(x_{-},\bar{\theta}) = h^{\dagger}(x,\theta,\bar{\theta}) \tag{2.28}$$

where

$$\begin{aligned} &\delta x_{+} = v(x_{+},\theta), \qquad \delta\theta = \lambda(x_{+},\theta) \qquad \text{functions of } x_{+} \text{ and } \theta \text{ only} \\ &\delta x_{-} = \bar{v}(x_{-},\bar{\theta}) \;\;, \quad \delta\bar{\theta} = \bar{\lambda}(x_{-},\bar{\theta}) \qquad \text{functions of } x_{-} \text{ and } \bar{\theta} \text{ only} \end{aligned} \tag{2.29}$$

Acting $\tilde{\sigma}_\mu^{\dot{\alpha}\alpha}\{\partial_\alpha^-, \bar{\partial}_{\dot{\alpha}}^+\} = -4i\partial_\mu$ to eq.(2.28) leads to

$$\partial_\mu h_\nu + \partial_\nu h_\mu = (D_\alpha \lambda^\alpha - \overline{D}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}})\ \eta_{\mu\nu} = \tfrac{1}{2}\partial \cdot h\ \eta_{\mu\nu} \tag{2.30}$$

Acting $\bar{\partial}_{\dot{\alpha}}^+$ to this gives

$$\partial_\mu(\lambda\sigma_\nu)_{\dot{\alpha}} + \partial_\nu(\lambda\sigma_\mu)_{\dot{\alpha}} = \tfrac{1}{2}\partial_\rho(\lambda\sigma^\rho)_{\dot{\alpha}}\eta_{\mu\nu} \tag{2.31}$$

and so

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \tfrac{1}{2}(\partial \cdot v)\eta_{\mu\nu} \tag{2.32}$$

Hence h, v, λ should be at most quadratic in x, and so we can put

$$\begin{aligned} v^\mu(x_+, \theta) &= v^\mu(\theta) + w^\mu{}_\nu(\theta)x_+^\nu + d(\theta)x_+^\mu + (\delta_\nu^\mu x_+^2 - 2x_+^\mu x_{+\nu})v^\nu(\theta) \\ \lambda^\alpha(x_+, \theta) &= \lambda^\alpha(\theta) + \lambda_\mu^\alpha(\theta)x_+^\mu + \lambda_{\mu\nu}^\alpha(\theta)(\eta^{\mu\nu}x_+^2 - 2x_+^\mu x_+^\nu) \end{aligned} \tag{2.33}$$

After substituting these expressions into the infinitesimal reality condition (2.28) and then by imposing the reality condition on the second, first and zeroth order terms in x successively we can derive the following most general solution of the infinitesimal reality condition, all the generators of continuous superconformal transformations, after a bit long tedious calculation.

$$v^\mu(x_+, \theta) = a^\mu + 2i\theta\sigma^\mu\bar{\xi} + w^\mu{}_\nu x_+^\nu + \lambda x_+^\mu - 2\theta\sigma^\mu(x_+ \cdot \tilde{\sigma})\tilde{\bar{\zeta}} + b^\mu x_+^2 - 2x_+^\mu b \cdot x_+ \tag{2.34}$$

$$\lambda^\alpha(x_+, \theta) = \xi^\alpha + \tfrac{1}{2}(\lambda + i\tau)\theta^\alpha + \tfrac{1}{4}w^{\mu\nu}(\theta\sigma_\mu\tilde{\sigma}_\nu)^\alpha + 2\theta^2\zeta^\alpha - i(\tilde{\bar{\zeta}}x_+ \cdot \tilde{\sigma})^\alpha + (\theta b \cdot \sigma x_+ \cdot \tilde{\sigma})^\alpha \tag{2.35}$$

where $\theta^2 = \theta^\alpha\theta_\alpha,\ \bar{\theta}^2 = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}},\ \tilde{\zeta} = \epsilon\zeta$, etc. By integration one can get the following finite superconformal transformations.

1. Supertranslations, $\hat{z} \oplus z$

$$x' = y + x - i\xi\sigma\bar{\theta} + i\theta\sigma\bar{\xi} \qquad \theta' = \xi + \theta \qquad \bar{\theta}' = \bar{\xi} + \bar{\theta} \tag{2.36}$$

   where $\hat{z} = y, \xi, \bar{\xi}$. One can check $g(\hat{z})g(z) = g(\hat{z} \oplus z)$.

2. Super Lorentz transformations, $L(w : z)$

$$x' = e^w x \qquad \theta' = \theta e^{\frac{1}{4}w^{\mu\nu}\sigma_\mu\tilde{\sigma}_\nu} \qquad \bar{\theta}' = e^{-\frac{1}{4}w^{\mu\nu}\tilde{\sigma}_\mu\sigma_\nu}\bar{\theta} \tag{2.37}$$

   where $(w)^\mu{}_\nu = w^\mu{}_\nu,\ w_{\mu\nu} = -w_{\nu\mu}$.

3. Superdilations, $d(\lambda : z)$

$$x' = |\lambda| x \qquad \theta' = \lambda^{\frac{1}{2}} \theta \qquad \bar{\theta}' = \bar{\lambda}^{\frac{1}{2}} \bar{\theta} \tag{2.38}$$

where $\lambda^{\frac{1}{2}}$ is an arbitrary complex number and $\bar{\lambda} = \lambda^{\dagger}$.

4. Superinversion, $i(z)$
Superinversion, $i(z)$, a superinversion-type transformation as the name indicates, is defined as [31]

$$x'_{\pm} = \frac{x_{\mp}}{x_{\mp}^2} \qquad \theta' = -i \frac{\tilde{\bar{\tilde{\theta}}}\; x_- \cdot \tilde{\sigma}}{x_-^2} \qquad \bar{\theta}' = i \frac{x_+ \cdot \tilde{\sigma} \tilde{\theta}}{x_+^2} \tag{2.39}$$

This definition implies $\Omega(i; z) = (x^2 + \theta^2 \bar{\theta}^2)^{-1}$ and $x' = x/(x^2 + \theta^2 \bar{\theta}^2)$.

Special superconformal transformation is defined by $i(\hat{z} \oplus i(z))$

$$\begin{aligned} x'_+ &= \frac{x_+ + b_- x_+^2 + 2\zeta \sigma x_+ \cdot \tilde{\sigma} \tilde{\theta}}{1 + 2x_+ \cdot b_- + x_+^2 b_-^2 + 4\zeta b_- \cdot \sigma x_+ \cdot \tilde{\sigma} \tilde{\theta} - 4\zeta \tilde{\theta} - 8\zeta^2 \theta^2} \\ \theta' &= \frac{\theta - \theta x_+ \cdot \sigma b_- \cdot \tilde{\sigma} + 4\zeta \theta^2 - i \tilde{\bar{\tilde{\zeta}}} x_+ \cdot \tilde{\sigma} - i \tilde{\bar{\tilde{\zeta}}} b_- \cdot \tilde{\sigma} x_+^2 + 4i\theta x_+ \cdot \sigma \bar{\zeta} \zeta}{1 + 2x_+ \cdot b_- + x_+^2 b_-^2 + 4\zeta b_- \cdot \sigma x_+ \cdot \tilde{\sigma} \tilde{\theta} - 4\zeta \tilde{\theta} - 8\zeta^2 \theta^2} \end{aligned} \tag{2.40}$$

where $\hat{z} = b, \zeta, \bar{\zeta}$. Infinitesimally this definition coincides with eqs.(2.34,2.35).
It is now clear that continuous superconformal transformations and superinversion-type transformations are one to one mapped by superinversion. From now on we will call $\{\oplus, L, d, i\}$ the fundamental elements of superconformal group - in the sense that any superconformal transformation can be generated by combining them.

2.2 Superconformal Group Representation

Under the successive superconformal transformations, $g' \circ g : z \xrightarrow{g} z' \xrightarrow{g'} z''$
using $x_+ - x_- = 2i\theta\sigma\bar{\theta}$ and chain rule, one can show that if both g' and g are continuous superconformal transformations

$$\frac{\partial^- \theta''^{\alpha}}{\partial \theta'^{\beta}} \frac{\partial^- \theta'^{\beta}}{\partial \theta^{\gamma}} = \frac{\partial^- \theta''^{\alpha}}{\partial \theta^{\gamma}}, \qquad \frac{\partial^+ \bar{\theta}''^{\dot{\alpha}}}{\partial \bar{\theta}'^{\dot{\beta}}} \frac{\partial^+ \bar{\theta}'^{\dot{\beta}}}{\partial \bar{\theta}^{\dot{\gamma}}} = \frac{\partial^+ \bar{\theta}''^{\dot{\alpha}}}{\partial \bar{\theta}^{\dot{\gamma}}} \tag{2.41}$$

if g', g are continuous superconformal and superinversion-type transformations respectively

$$\frac{\partial^+ \bar{\theta}''^{\dot{\alpha}}}{\partial \bar{\theta}'^{\dot{\beta}}} \frac{\partial^- \bar{\theta}'^{\dot{\beta}}}{\partial \theta^{\alpha}} = \frac{\partial^- \bar{\theta}''^{\dot{\alpha}}}{\partial \theta^{\alpha}}, \qquad \frac{\partial^- \theta''^{\alpha}}{\partial \theta'^{\beta}} \frac{\partial^+ \theta'^{\beta}}{\partial \bar{\theta}^{\dot{\alpha}}} = \frac{\partial^+ \theta''^{\alpha}}{\partial \bar{\theta}^{\dot{\alpha}}} \tag{2.42}$$

if g', g are superinversion-type and continuous superconformal transformations respectively

$$\frac{\partial^{+}\theta''^{\alpha}}{\partial\bar{\theta}'^{\dot{\alpha}}}\frac{\partial^{+}\bar{\theta}'^{\dot{\alpha}}}{\partial\bar{\theta}^{\dot{\beta}}}=\frac{\partial^{+}\theta''^{\alpha}}{\partial\bar{\theta}^{\dot{\beta}}}, \qquad \frac{\partial^{-}\bar{\theta}''^{\dot{\alpha}}}{\partial\theta'^{\alpha}}\frac{\partial^{-}\theta'^{\alpha}}{\partial\theta^{\beta}}=\frac{\partial^{-}\bar{\theta}''^{\dot{\alpha}}}{\partial\theta^{\beta}} \tag{2.43}$$

if both g' and g are superinversion-type transformations

$$\frac{\partial^{+}\theta''^{\alpha}}{\partial\bar{\theta}'^{\dot{\alpha}}}\frac{\partial^{-}\bar{\theta}'^{\dot{\alpha}}}{\partial\theta^{\beta}}=\frac{\partial^{-}\theta''^{\alpha}}{\partial\theta^{\beta}}, \qquad \frac{\partial^{-}\bar{\theta}''^{\dot{\alpha}}}{\partial\theta'^{\alpha}}\frac{\partial^{+}\theta'^{\alpha}}{\partial\bar{\theta}^{\dot{\beta}}}=\frac{\partial^{+}\bar{\theta}''^{\dot{\alpha}}}{\partial\bar{\theta}^{\dot{\beta}}} \tag{2.44}$$

for any superconformal transformation g' and g

$$A^{\mu}{}_{\nu}(g';z')A^{\nu}{}_{\rho}(g;z)=A^{\mu}{}_{\rho}(g'\circ g;z) \tag{2.45}$$

Thus the followings are representations of superconformal group.

- For continuous superconformal transformations

$$A(g;z) \qquad \frac{\partial^{-}\theta'}{\partial\theta} \qquad \frac{\partial^{+}\bar{\theta}'}{\partial\bar{\theta}} \tag{2.46}$$

- For superinversion-type transformations

$$A(g;z) \qquad \frac{\partial^{-}\bar{\theta}'}{\partial\theta} \qquad \frac{\partial^{+}\theta'}{\partial\bar{\theta}} \tag{2.47}$$

These are compactified representations of superconformal group from 6×6 fundamental representation to 4×4 or 2×2. $A^{\mu}{}_{\nu}(g;z)$ is the superconformal representation for the infinitesimal supersymmetric interval w^{μ}

$$\mathsf{w}'^{\mu}=A^{\mu}{}_{\nu}(g;z)\mathsf{w}^{\nu} \tag{2.48}$$

and $\frac{\partial^{-}\theta'}{\partial\theta}$, $\frac{\partial^{+}\bar{\theta}'}{\partial\bar{\theta}}$, $\frac{\partial^{-}\bar{\theta}'}{\partial\theta}$, $\frac{\partial^{+}\theta'}{\partial\bar{\theta}}$ are the superconformal representations for the left invariant derivatives $D_{\alpha}=\partial^{-}_{\alpha}$, $\overline{D}_{\dot{\alpha}}=-\bar{\partial}^{+}_{\dot{\alpha}}$

$$\left.\begin{aligned} D_{\alpha}&=\frac{\partial^{-}\theta'^{\beta}}{\partial\theta^{\alpha}}D'_{\beta} \\ \overline{D}_{\dot{\alpha}}&=\frac{\partial^{+}\bar{\theta}'^{\dot{\beta}}}{\partial\bar{\theta}^{\dot{\alpha}}}\overline{D}'_{\dot{\beta}} \end{aligned}\right\}\quad \text{for continuous superconformal transformation} \tag{2.49}$$

$$\left.\begin{aligned} D_{\alpha}&=-\frac{\partial^{-}\bar{\theta}'^{\dot{\alpha}}}{\partial\theta^{\alpha}}\overline{D}'_{\dot{\alpha}} \\ \overline{D}_{\dot{\alpha}}&=-\frac{\partial^{+}\theta'^{\alpha}}{\partial\bar{\theta}^{\dot{\alpha}}}D'_{\alpha} \end{aligned}\right\}\quad \text{for superinversion-type transformation} \tag{2.50}$$

One can also get

$$\partial_\mu = A^\nu{}_\mu(g;z)\partial'_\nu + \frac{\partial\theta'^\alpha}{\partial x^\mu}D'_\alpha - \frac{\partial\bar{\theta}'^{\dot{\alpha}}}{\partial x^\mu}\overline{D}'_{\dot{\alpha}} \tag{2.51}$$

For the fundamental elements of superconformal group we have

1. Supertranslations

$$A^\mu{}_\nu = \delta^\mu{}_\nu \qquad \frac{\partial^-\theta'^\alpha}{\partial\theta^\beta} = \delta^\alpha_\beta \qquad \frac{\partial^+\bar{\theta}'^{\dot{\alpha}}}{\partial\bar{\theta}^{\dot{\beta}}} = \delta^{\dot{\alpha}}_{\dot{\beta}} \tag{2.52}$$

2. Super Lorentz transformations

$$A^\mu{}_\nu = (e^w)^\mu{}_\nu \qquad \frac{\partial^-\theta'^\alpha}{\partial\theta^\beta} = (e^{\frac{1}{4}w^{\mu\nu}\sigma_\mu\tilde{\sigma}_\nu})_\beta{}^\alpha \qquad \frac{\partial^+\bar{\theta}'^{\dot{\alpha}}}{\partial\bar{\theta}^{\dot{\beta}}} = (e^{-\frac{1}{4}w^{\mu\nu}\tilde{\sigma}_\mu\sigma_\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \tag{2.53}$$

3. Superdilations

$$A^\mu{}_\nu = |\lambda|\delta^\mu{}_\nu \qquad \frac{\partial^-\theta'^\alpha}{\partial\theta^\beta} = \lambda^{\frac{1}{2}}\delta^\alpha_\beta \qquad \frac{\partial^+\bar{\theta}'^{\dot{\alpha}}}{\partial\bar{\theta}^{\dot{\beta}}} = \bar{\lambda}^{\frac{1}{2}}\delta^{\dot{\alpha}}_{\dot{\beta}} \tag{2.54}$$

4. Superinversion

$$\begin{aligned} A^\mu{}_\nu &= \frac{1}{x^2+\theta^2\bar{\theta}^2}\left(\frac{x^2-\theta^2\bar{\theta}^2}{x^2+\theta^2\bar{\theta}^2}\delta^\mu{}_\nu - 2\frac{x^\mu x_\nu}{x^2+\theta^2\bar{\theta}^2} + 2\epsilon^\mu{}_{\nu\lambda\kappa}\frac{\theta\sigma^\lambda\bar{\theta}}{x^2}x^\kappa\right) \\ \frac{\partial^-\bar{\theta}'_{\dot{\alpha}}}{\partial\theta^\alpha} &= -i\frac{x_+\cdot\sigma_{\alpha\dot{\alpha}}}{x_-^2} \qquad\qquad \frac{\partial^+\theta'_\alpha}{\partial\bar{\theta}^{\dot{\alpha}}} = i\frac{x_-\cdot\sigma_{\alpha\dot{\alpha}}}{x_+^2} \end{aligned} \tag{2.55}$$

From the property of superdeterminant one can show that for continuous superconformal transformation

$$\begin{aligned} \mathrm{sdet}\begin{pmatrix} \frac{\partial x'_+}{\partial x} & \frac{\partial\theta'}{\partial x} \\ \frac{\partial^+ x'_+}{\partial\theta} & \frac{\partial^+\theta'}{\partial\theta} \end{pmatrix} &= \mathrm{sdet}\begin{pmatrix} A & 0 \\ 0 & \frac{\partial^-\theta'}{\partial\theta} \end{pmatrix} \overset{\text{shortly}}{=} \mathrm{sdet}_c \\ \mathrm{sdet}\begin{pmatrix} \frac{\partial x'_-}{\partial x} & \frac{\partial\bar{\theta}'}{\partial x} \\ \frac{\partial^- x'_-}{\partial\theta} & \frac{\partial^-\bar{\theta}'}{\partial\theta} \end{pmatrix} &= \mathrm{sdet}\begin{pmatrix} A & 0 \\ 0 & \frac{\partial^+\bar{\theta}'}{\partial\theta} \end{pmatrix} \overset{\text{shortly}}{=} \overline{\mathrm{sdet}_c} \end{aligned} \tag{2.56}$$

for superinversion-type transformation

$$\begin{aligned}
\mathrm{sdet}\begin{pmatrix} \frac{\partial x'_-}{\partial x} & \frac{\partial \bar\theta'}{\partial x} \\ \frac{\partial^+ x'_-}{\partial \theta} & \frac{\partial^+ \bar\theta'}{\partial \theta} \end{pmatrix} &= \mathrm{sdet}\begin{pmatrix} A & 0 \\ 0 & \frac{\partial^- \bar\theta'}{\partial \theta} \end{pmatrix} \overset{\text{shortly}}{=} \mathrm{sdet}_i \\
\mathrm{sdet}\begin{pmatrix} \frac{\partial x'_+}{\partial x} & \frac{\partial \theta'}{\partial x} \\ \frac{\partial^- x'_+}{\partial \bar\theta} & \frac{\partial^- \theta'}{\partial \bar\theta} \end{pmatrix} &= \mathrm{sdet}\begin{pmatrix} A & 0 \\ 0 & \frac{\partial^+ \theta'}{\partial \bar\theta} \end{pmatrix} \overset{\text{shortly}}{=} \overline{\mathrm{sdet}}_i
\end{aligned} \tag{2.57}$$

for any superconformal transformation

$$\mathrm{sdet} = (\overline{\mathrm{sdet}})^\dagger \tag{2.58}$$

The following formulae can be verified for each fundamental element of superconformal group by direct calculation and from the fact that $A^\mu{}_\nu$, $\frac{\partial^- \theta'^\alpha}{\partial \theta^\beta}$, $\frac{\partial^+ \bar\theta'^{\dot\alpha}}{\partial \bar\theta^{\dot\beta}}$ are representations of superconformal group, they can be generalized to any superconformal transformation.

- For continuous superconformal transformation

$$A^t(g;z)\eta A(g;z) = (\mathrm{sdet}_c\overline{\mathrm{sdet}}_c)^{\frac{1}{3}}\eta \qquad \det A = (\mathrm{sdet}_c\overline{\mathrm{sdet}}_c)^{\frac{2}{3}} \tag{2.59}$$

$$\frac{\partial^- \theta'^\gamma}{\partial \theta^\alpha}\frac{\partial^- \theta'^\delta}{\partial \theta^\beta}\epsilon_{\gamma\delta} = \frac{(\mathrm{sdet}_c\overline{\mathrm{sdet}}_c)^{\frac{2}{3}}}{\mathrm{sdet}_c}\epsilon_{\alpha\beta} \qquad \det\left(\frac{\partial^- \theta'}{\partial \theta}\right) = \frac{(\mathrm{sdet}_c\overline{\mathrm{sdet}}_c)^{\frac{2}{3}}}{\mathrm{sdet}_c} \tag{2.60}$$

$$\frac{\partial^+ \bar\theta'^{\dot\gamma}}{\partial \bar\theta^{\dot\alpha}}\frac{\partial^+ \bar\theta'^{\dot\delta}}{\partial \bar\theta^{\dot\beta}}\epsilon_{\dot\gamma\dot\delta} = \frac{(\mathrm{sdet}_c\overline{\mathrm{sdet}}_c)^{\frac{2}{3}}}{\overline{\mathrm{sdet}}_c}\epsilon_{\dot\alpha\dot\beta} \qquad \det\left(\frac{\partial^+ \bar\theta'}{\partial \bar\theta}\right) = \frac{(\mathrm{sdet}_c\overline{\mathrm{sdet}}_c)^{\frac{2}{3}}}{\overline{\mathrm{sdet}}_c} \tag{2.61}$$

$$\frac{\partial^- \theta'^\beta}{\partial \theta^\alpha}\sigma^\mu_{\beta\dot\beta}\frac{\partial^+ \bar\theta'^{\dot\beta}}{\partial \bar\theta^{\dot\alpha}} = A^\mu{}_\nu(g;z)\sigma^\nu_{\alpha\dot\alpha} \tag{2.62}$$

- For superinversion-type transformation

$$A^t(g;z)\eta A(g;z) = (\mathrm{sdet}_i\overline{\mathrm{sdet}}_i)^{\frac{1}{3}}\eta \qquad \det A = -(\mathrm{sdet}_i\overline{\mathrm{sdet}}_i)^{\frac{2}{3}} \tag{2.63}$$

$$\frac{\partial^- \bar\theta'^{\dot\alpha}}{\partial \theta^\alpha}\epsilon_{\dot\alpha\dot\beta}\frac{\partial^- \bar\theta'^{\dot\beta}}{\partial \theta^\beta} = \frac{(\mathrm{sdet}_i\overline{\mathrm{sdet}}_i)^{\frac{2}{3}}}{\mathrm{sdet}_i}\epsilon_{\alpha\beta} \qquad \det\left(\frac{\partial^- \bar\theta'}{\partial \theta}\right) = -\frac{(\mathrm{sdet}_i\overline{\mathrm{sdet}}_i)^{\frac{2}{3}}}{\mathrm{sdet}_i} \tag{2.64}$$

$$\frac{\partial^+\theta'^\alpha}{\partial\bar\theta^{\dot\alpha}}\epsilon_{\alpha\beta}\frac{\partial^+\theta'^\beta}{\partial\bar\theta^{\dot\beta}} = \frac{(\mathrm{sdet}_i\overline{\mathrm{sdet}_i})^{\frac{2}{3}}}{\overline{\mathrm{sdet}_i}}\epsilon_{\dot\alpha\dot\beta} \qquad \det\left(\frac{\partial^+\theta'}{\partial\bar\theta}\right) = -\frac{(\mathrm{sdet}_i\overline{\mathrm{sdet}_i})^{\frac{2}{3}}}{\overline{\mathrm{sdet}_i}} \tag{2.65}$$

$$\frac{\partial^-\bar\theta'_{\dot\beta}}{\partial\theta^\alpha}\tilde\sigma^{\mu\dot\beta\beta}\frac{\partial^+\theta'_\beta}{\partial\bar\theta^{\dot\alpha}} = A^\mu{}_\nu(g;z)\sigma^\nu_{\alpha\dot\alpha} \tag{2.66}$$

We identify the local scale factor

$$\Omega(g;z) = (\det A(g;z))^{\frac{1}{4}} = (\mathrm{sdet}\overline{\mathrm{sdet}})^{\frac{1}{6}} \tag{2.67}$$

We define a local Lorentz transformation, $\mathcal{R}(g;z)$ for any superconformal transformation, g by

$$\mathcal{R}^\mu{}_\nu(g;z) = \Omega^{-1}(g;z)A^\mu{}_\nu(g;z) \qquad\qquad \mathcal{R}^t\eta\mathcal{R} = \eta \tag{2.68}$$

Surely $\mathcal{R}(g;z)$ is also a representation of superconformal group. Specially for superinversion we denote

$$I^\mu{}_\nu(z) = \mathcal{R}^\mu{}_\nu(i;z) = \frac{x^2-\theta^2\bar\theta^2}{x^2+\theta^2\bar\theta^2}\delta^\mu{}_\nu - 2\frac{x^\mu x_\nu}{x^2+\theta^2\bar\theta^2} + 2\epsilon^\mu{}_{\nu\lambda\kappa}\frac{\theta\sigma^\lambda\bar\theta}{x^2}x^\kappa \tag{2.69}$$

This expression will be frequently recalled later. It is worth to note $I(i(z)) = I^{-1}(z) = I(-z)$.

3 Superconformal Transformation Rules for Superfields

In this section we study the superconformal transformation rules for the chiral/anti-chiral superfields and the supercurrents in Wess-Zumino model and vector superfield theory respectively.

3.1 In Wess-Zumino Model

In Wess-Zumino model

$$-\tfrac{1}{8}\!\int\!\int d^4x_+d^2\theta\ \Phi\overline{D}^2\overline{\Phi} = \int d^4x\ \left(\partial_\mu\phi\partial^\mu\bar\phi + 2i\psi\sigma^\mu\partial_\mu\bar\psi\right) \tag{3.1}$$

chiral superfield

$$\Phi(x_+,\theta) = \phi(x_+) + 2\theta^\alpha\psi_\alpha(x_+) + \theta^2F(x_+) \tag{3.2}$$

and anti-chiral superfield, $\overline{\Phi} = \Phi^\dagger$ transform under superconformal transformation $g : z \to z'$, as

- for continuous superconformal transformation

$$\begin{aligned} \Phi(x_+,\theta) &\rightarrow \mathrm{sdet}_c^{\frac{1}{3}}\Phi(x'_+,\theta') \\ \overline{\Phi}(x_-,\bar\theta) &\rightarrow \overline{\mathrm{sdet}}_c^{\frac{1}{3}}\overline{\Phi}(x'_-,\bar\theta') \end{aligned} \tag{3.3}$$

- for superinversion-type transformation

$$\begin{aligned} \Phi(x_+,\theta) &\rightarrow \mathrm{sdet}_i^{\frac{1}{3}}\overline{\Phi}(x'_-,\bar\theta') \\ \overline{\Phi}(x_-,\bar\theta) &\rightarrow \overline{\mathrm{sdet}}_i^{\frac{1}{3}}\Phi(x'_+,\theta') \end{aligned} \tag{3.4}$$

These transformations ensure that every component field transforms properly [32, 33] so that the Lagrangian, $\partial_\mu\phi\partial^\mu\bar\phi + 2i\psi\sigma^\mu\partial_\mu\bar\psi$ transforms as a scalar density with weight 1 and so the action does not change. This gives the superconformal invariance of the correlation functions. One can easily check this for each element of superconformal group and from the fact that superdeterminant is a representation of superconformal group, they hold for any superconformal transformation.
The supercurrent, $J_{\alpha\dot\alpha}$, in Wess-Zumino model is [34]

$$J_{\alpha\dot\alpha} = D_\alpha\Phi\overline{D}_{\dot\alpha}\overline{\Phi} + 2i\Phi\overleftrightarrow{\partial}_{\alpha\dot\alpha}\overline{\Phi} \tag{3.5}$$

Equations of motion $\overline{D}^2\overline{\Phi} = 0, \quad D^2\Phi = 0$ make the supercurrent conserved

$$D^\alpha J_{\alpha\dot\alpha} = 0 \tag{3.6}$$

Correlation functions of the supercurrent also satisfy the conservation equation. The typical term of a correlation function, $\langle\cdots J_{\alpha\dot\alpha}\cdots A\cdots B\cdots\rangle$ is

$$D_\alpha\langle\Phi A\rangle\overline{D}_{\dot\alpha}\langle\overline{\Phi}B\rangle + 2i(-1)^{\#A}\langle\Phi A\rangle \overleftrightarrow{\partial}_{\alpha\dot\alpha} \langle\overline{\Phi}B\rangle \tag{3.7}$$

where $\#A$ is $+1$ for bosonic A and -1 for fermionic A. With chirality, $\overline{D}_{\dot\alpha}\langle\Phi A\rangle = 0$ and the equation of motion, $D^2\langle\Phi A\rangle = 0$, one can show that eq.(3.7) satisfies the conservation equation.
The power one third appearing in eqs.(3.3,3.4) enables us to add a superpotential term, $g\Phi^3$, to the super Lagrangian still maintaining the superconformal symmetry to get an interacting superconformal field theory, but the supercurrent is not conserved any more.

$$D^\alpha J_{\alpha\dot\alpha} \propto g\overline{\Phi}^2\overline{D}_{\dot\alpha}\overline{\Phi} \tag{3.8}$$

The only non-vanishing two-point correlation function of chiral/anti-chiral massless free superfields is

$$\langle\Phi(x_{1+},\theta_1)\overline{\Phi}(x_{2-},\bar{\theta}_2)\rangle = \frac{c}{(x_{1+}-x_{2-}-2i\theta_1\sigma\bar{\theta}_2)^2} = \frac{c}{\left(x_{12}+i\theta_{12}\sigma\bar{\theta}_{12}\right)^2} \tag{3.9}$$

where $c = -\frac{1}{2\pi^2}$. Under superconformal transformation the chiral superfield transforms as $\Phi(z) \to \Phi(z')$ or $\overline{\Phi}(z')$ up to a scale factor, hence eq.(3.9) implies that the infinitesimal interval length, w^2 is invariant up to a scale factor under superconformal transformation. Of course, this is consistent with the definition of superconformal group (2.7). Superconformal symmetry of the two-point correlation function gives

$$\begin{aligned} x'^2_{12\pm} &= \mathrm{sdet}_{c1}^{\frac{1}{3}}\overline{\mathrm{sdet}}_{c2}^{\frac{1}{3}}\; x^2_{12\pm} \qquad &&\text{for continuous superconformal transformation} \\ x'^2_{12\pm} &= \mathrm{sdet}_{i1}^{\frac{1}{3}}\overline{\mathrm{sdet}}_{i2}^{\frac{1}{3}}\; x^2_{12\mp} \qquad &&\text{for superinversion-type transformation} \end{aligned} \tag{3.10}$$

Multiplying x^2_{12+} and x^2_{12-} gives

$$x'^2_{12} + \theta'^2_{12}\bar{\theta}'^2_{12} = \Omega(g;z_1)\Omega(g;z_2)\left(x^2_{12}+\theta^2_{12}\bar{\theta}^2_{12}\right) \tag{3.11}$$

Using the superconformal transformation rules for the chiral/anti-chiral superfield and the left invariant derivatives one can show that the supercurrent in Wess-Zumino model transforms under continuous superconformal transformation as

$$\begin{aligned} J_{\alpha\dot{\alpha}}(z) \rightarrow\;\; & s\bar{s}\tfrac{\partial^-\theta'^\beta}{\partial\theta^\alpha}\tfrac{\partial^+\bar{\theta}'^{\dot{\beta}}}{\partial\bar{\theta}^{\dot{\alpha}}}J_{\beta\dot{\beta}}(z') + (D_\alpha s\overline{D}_{\dot{\alpha}}\bar{s} + 2is\overset{\leftrightarrow}{\partial}_{\alpha\dot{\alpha}}\bar{s})\Phi'\overline{\Phi}' \\ & +(D_\alpha s\tfrac{\partial^+\bar{\theta}'^{\dot{\beta}}}{\partial\bar{\theta}^{\dot{\alpha}}} - 2is\partial_{\alpha\dot{\alpha}}\bar{\theta}'^{\dot{\beta}})\bar{s}\Phi'\overline{D}'_{\dot{\beta}}\overline{\Phi}' - (\overline{D}_{\dot{\alpha}}\bar{s}\tfrac{\partial^-\theta'^\beta}{\partial\theta^\alpha} + 2i\bar{s}\partial_{\alpha\dot{\alpha}}\theta'^\beta)s\overline{\Phi}'D'_\beta\Phi' \end{aligned} \tag{3.12}$$

where $s = \mathrm{sdet}_c^{\frac{1}{3}}$, $\bar{s} = \overline{\mathrm{sdet}}_c^{\frac{1}{3}}$, and under superinversion-type transformation

$$\begin{aligned} J_{\alpha\dot{\alpha}}(z) \rightarrow\;\; & -s\bar{s}\tfrac{\partial^-\bar{\theta}'^{\dot{\beta}}}{\partial\theta^\alpha}\tfrac{\partial^+\theta'^\beta}{\partial\bar{\theta}^{\dot{\alpha}}}J_{\beta\dot{\beta}}(z') + (D_\alpha s\overline{D}_{\dot{\alpha}}\bar{s} + 2is\overset{\leftrightarrow}{\partial}_{\alpha\dot{\alpha}}\bar{s})\Phi'\overline{\Phi}' \\ & -(D_\alpha s\tfrac{\partial^+\theta'^\beta}{\partial\bar{\theta}^{\dot{\alpha}}} - 2is\partial_{\alpha\dot{\alpha}}\theta'^\beta)\bar{s}\overline{\Phi}'D'_\beta\overline{\Phi}' + (\overline{D}_{\dot{\alpha}}\bar{s}\tfrac{\partial^-\bar{\theta}'^{\dot{\beta}}}{\partial\theta^\alpha} + 2i\bar{s}\partial_{\alpha\dot{\alpha}}\bar{\theta}'^{\dot{\beta}})s\Phi'\overline{D}'_{\dot{\beta}}\overline{\Phi}' \end{aligned} \tag{3.13}$$

where $s = \mathrm{sdet}_i^{\frac{1}{3}}$, $\bar{s} = \overline{\mathrm{sdet}}_i^{\frac{1}{3}}$. However one can also check[3] that for each fundamental element of superconformal group except superinversion

$$\begin{gathered} D_\alpha s \tfrac{\partial^+ \bar{\theta}'^{\dot{\beta}}}{\partial \bar{\theta}^{\dot{\alpha}}} - 2is\partial_{\alpha\dot{\alpha}}\bar{\theta}'^{\dot{\beta}} = 0 \qquad \overline{D}_{\dot{\alpha}} \bar{s} \tfrac{\partial^- \theta'^{\beta}}{\partial \theta^{\alpha}} + 2i\bar{s}\partial_{\alpha\dot{\alpha}}\theta'^{\beta} = 0 \\ D_\alpha s \overline{D}_{\dot{\alpha}} \bar{s} + 2is \overleftrightarrow{\partial}_{\alpha\dot{\alpha}} \bar{s} = 0 \end{gathered} \tag{3.14}$$

and for superinversion

$$\begin{gathered} D_\alpha s \tfrac{\partial^+ \theta'^{\beta}}{\partial \bar{\theta}^{\dot{\alpha}}} - 2is\partial_{\alpha\dot{\alpha}}\theta'^{\beta} = 0 \qquad \overline{D}_{\dot{\alpha}} \bar{s} \tfrac{\partial^- \bar{\theta}'^{\dot{\beta}}}{\partial \theta^{\alpha}} + 2i\bar{s}\partial_{\alpha\dot{\alpha}}\bar{\theta}'^{\dot{\beta}} = 0 \\ D_\alpha s \overline{D}_{\dot{\alpha}} \bar{s} + 2is \overleftrightarrow{\partial}_{\alpha\dot{\alpha}} \bar{s} = 0 \end{gathered} \tag{3.15}$$

This gives the following superconformal transformation rule for the supercurrent in Wess-Zumino model[4]

$$J^\mu(z) \rightarrow \pm_g \Omega^3(g;z) \mathcal{R}^{-1\mu}{}_\nu(g;z) J^\nu(z') \tag{3.16}$$

where $J^\mu = -\frac{1}{2}\tilde{\sigma}^{\mu\dot{\alpha}\alpha} J_{\alpha\dot{\alpha}}$, $J_{\alpha\dot{\alpha}} = \sigma_{\mu\alpha\dot{\alpha}} J^\mu$ and

$$\pm_g = \begin{cases} +1 & \text{for continuous } g \\ -1 & \text{for superinversion-type } g \end{cases} \tag{3.17}$$

3.2 In Vector Superfield Theory

In vector superfield theory

$$-\tfrac{1}{8} \int d^4x \, \left(\int d^2\theta \; W^\alpha W_\alpha + \int d^2\bar{\theta} \; \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}\right) = \int d^4x \, \left(\tfrac{1}{4} f^{\mu\nu} f_{\mu\nu} + i\lambda\sigma^\mu\partial_\mu\bar{\lambda}\right) \tag{3.18}$$

where $f_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$. Fermionic chiral superfield, W_α

$$W_\alpha(x_+,\theta) = -i\lambda_\alpha(x_+) + \{D(x_+)\delta_\alpha^\beta - \tfrac{1}{2} i(\sigma^\mu\tilde{\sigma}^\nu)_\alpha{}^\beta f_{\mu\nu}(x_+)\}\theta_\beta + \theta^2 \partial_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(x_+) \tag{3.19}$$

and anti-chiral superfield, $\overline{W}_{\dot{\alpha}} = (W_\alpha)^\dagger$ transform under continuous superconformal transformation as

$$\begin{gathered} W_\alpha(x_+,\theta) \rightarrow \frac{\mathrm{sdet}_c^{\frac{2}{3}}}{\overline{\mathrm{sdet}}_c^{\frac{1}{3}}} \frac{\partial^- \theta'^{\beta}}{\partial \theta^\alpha} W_\beta(x'_+,\theta') \\ \overline{W}_{\dot{\alpha}}(x_-,\bar{\theta}) \rightarrow \frac{\overline{\mathrm{sdet}}_c^{\frac{2}{3}}}{\mathrm{sdet}_c^{\frac{1}{3}}} \frac{\partial^+ \bar{\theta}'^{\dot{\beta}}}{\partial \bar{\theta}^{\dot{\alpha}}} \overline{W}_{\dot{\beta}}(x'_-,\bar{\theta}') \end{gathered} \tag{3.20}$$

[3]In fact, considering the successive superconformal transformations one can show that eqs.(3.14,3.15) hold for any continuous/superinversion-type transformation.

[4]This superconformal transformation rule is valid for any superconformal transformation, since $\pm_g\Omega^3(g;z)\mathcal{R}^{-1\mu}{}_\nu(g;z)$ is a representation of superconformal group.

and under superinversion-type transformation

$$
\begin{aligned}
W_\alpha(x_+,\theta) &\to \frac{\mathrm{sdet}_i^{\frac{2}{3}}}{\overline{\mathrm{sdet}}_i^{\frac{1}{3}}}\frac{\partial^-\bar\theta'^{\dot\alpha}}{\partial\theta^\alpha}\overline{W}_{\dot\alpha}(x'_-,\bar\theta') \\
\overline{W}_{\dot\alpha}(x_-,\bar\theta) &\to \frac{\overline{\mathrm{sdet}}_i^{\frac{2}{3}}}{\mathrm{sdet}_i^{\frac{1}{3}}}\frac{\partial^+\theta'^{\alpha}}{\partial\bar\theta^{\dot\alpha}}W_\alpha(x'_+,\theta')
\end{aligned}
\tag{3.21}
$$

One can check that these transformations leave the action invariant.
The supercurrent $\tilde J_{\alpha\dot\alpha}$ in vector superfield theory is

$$
\tilde J_{\alpha\dot\alpha} = W_\alpha \overline{W}_{\dot\alpha} \tag{3.22}
$$

Equation of motion $D^\alpha W_\alpha = 0$ makes the supercurrent conserved, $D^\alpha \tilde J_{\alpha\dot\alpha} = 0$. The supercurrent transforms under superconformal transformation as

$$
\tilde J^\mu(z) \to \pm_g \Omega^3(g;z)\mathcal{R}^{-1\mu}{}_\nu(g;z)\tilde J^\nu(z') \tag{3.23}
$$

4 Superconformal Invariance of Correlation Function

From the result of the previous section we have superconformal transformation rule for Konishi current, $\Phi(x_+,\theta)\overline{\Phi}(x_-,\bar\theta)$

$$
\Phi(x_+,\theta)\overline{\Phi}(x_-,\bar\theta) \ \to \Omega^2(g;z)\Phi(x'_+,\theta')\overline{\Phi}(x'_-,\bar\theta') \tag{4.1}
$$

Keeping this and the superconformal transformation rule for the supercurrents (3.16,3.23) in our mind, in this section we study the correlation functions of superfields in a group theoretical way. We write superfield as $\Psi^i(z)$ and assume that $\Psi^i(z)$ transforms under superconformal transformation, $g: z \longrightarrow z'$, as[5]

$$
\Psi^i(z) \to \Omega^\eta(g;z) D^i{}_j(\mathcal{R}^{-1}(g;z))\Psi^j(z') \tag{4.2}
$$

where η is the scale dimension of the superfield and $D^i{}_j(\mathcal{R}^{-1}(g;z))$ is a representation for the local Lorentz transformation, $\mathcal{R}^{-1}(g;z)$ and so under the successive superconformal transformations, $z \xrightarrow{g} z' \xrightarrow{g'} z''$

$$
D^i{}_j(\mathcal{R}^{-1}(g;z))D^j{}_k(\mathcal{R}^{-1}(g';z')) = D^i{}_k(\mathcal{R}^{-1}(g'\circ g;z)) \tag{4.3}
$$

[5]Here we focus on only real valued scale factor and exclude so called R-factor.

Correlation function has superconformal symmetry

$$\begin{aligned}
&\langle \Psi_1^{i_1}(z_1)\Psi_2^{i_2}(z_2)\cdots\Psi_n^{i_n}(z_n)\rangle \\
&= \prod_{a=1}^n \Omega^{\eta_a}(g;z_a) D_a{}^{i_a}{}_{j_a}(\mathcal{R}^{-1}(g;z_a))\langle \Psi_1^{j_1}(z_1')\Psi_2^{j_2}(z_2')\cdots\Psi_n^{j_n}(z_n')\rangle
\end{aligned} \tag{4.4}$$

where the subscript, a of Ψ_a^i denotes the type of the superfield, i.e. scalar superfield, supervector field, etc.

4.1 Two-point Correlation Function of General Superfields

Here we will show that if two superfields have different scale dimensions then the two-point function of them vanishes, and if they belong to a same type and so the scale dimensions are equal then two-point function is of the following general form

$$\langle \Psi^i(z_1)\Psi^j(z_2)\rangle = \frac{D^i{}_{i'}(I^{-1}(z_{12}))\mathcal{H}^{i'j}}{(x_{12}^2+\theta_{12}^2\bar{\theta}_{12}^2)^\eta} \tag{4.5}$$

where $z_{12} = -z_2 \oplus z_1$, η is the scale dimension of the superfield and $\mathcal{H}^{ij}$ is a constant matrix which satisfies the superconformal symmetric condition

$$\mathcal{H}^{ij} = D^i{}_{i'}(L)D^j{}_{j'}(L)\mathcal{H}^{i'j'} \tag{4.6}$$

where L is an arbitrary Lorentz transformation.
proof
Without loss of generality, using the supertranslational invariance, we can put the two-point function as

$$\langle \Psi_1^{i_1}(z_1)\Psi_2^{i_2}(z_2)\rangle = \frac{D_{1i'}^{i}(I^{-1}(z_{12}))H^{i'j}(i(z_{12}))}{(x_{12}^2+\theta_{12}^2\bar{\theta}_{12}^2)^{\frac{\eta_1+\eta_2}{2}}} \tag{4.7}$$

For any superconformal transformation, $g: z \to z'$, one may show (see appendix A for our derivation)

$$\mathcal{R}^{-1}(g;z_2)I(z_{12}')\mathcal{R}(g;z_1) = I(z_{12}) \tag{4.8}$$

where $z_{12}' = -z_2' \oplus z_1'$. By virtue of this relation and eq.(3.11), the superconformal invariance of the correlation function (4.4) leads to

$$H^{ij}(i(z_{12}')) = \left(\frac{\Omega(g;z_2)}{\Omega(g;z_1)}\right)^{\frac{\eta_1-\eta_2}{2}} D_{1i'}^{i}(\mathcal{R}(g;z_2))D_{2j'}^{j}(\mathcal{R}(g;z_2))H^{i'j'}(i(z_{12})) \tag{4.9}$$

Let us consider a superconformal transformation, $\tilde{g} : z \to z'$ defined by

$$\tilde{g}(z) = z'_2 \oplus i(\tilde{z} \oplus i(-z_2 \oplus z)) \tag{4.10}$$

where $\tilde{g}(z_2) = z'_2$ and $\tilde{z}$ are arbitrary. This definition gives $i(z'_{12}) = \tilde{z} \oplus i(z_{12})$ and from the explicit form of special superconformal transformation (2.40) one can easily check that $\mathcal{R}^{\mu}{}_{\nu}(\tilde{g}; z_2) = \delta^{\mu}{}_{\nu}$, $\Omega(\tilde{g}; z_2) = 1$. On the other hand, we have

$$\Omega(\tilde{g}; z_1) = \Omega(i; i(z'_{12}))\Omega(i; z_{12}) = \frac{x'^2_{12} + \theta'^2_{12}\bar{\theta}'^2_{12}}{x^2_{12} + \theta^2_{12}\bar{\theta}^2_{12}} \tag{4.11}$$

Now we choose $\tilde{z} = i(z_o) \oplus -i(z_{12})$ to get $z'_{12} = z_o$, where z_o is arbitrary but fixed. Then eq.(4.9) becomes

$$H^{ij}(i(z_{12})) = \frac{\mathcal{H}^{ij}}{(x^2_{12} + \theta^2_{12}\bar{\theta}^2_{12})^{\frac{\eta_1 - \eta_2}{2}}} \tag{4.12}$$

where $\mathcal{H}^{ij} = (x^2_o + \theta^2_o\bar{\theta}^2_o)^{\frac{\eta_1 - \eta_2}{2}} H^{ij}(i(z_o))$ is constant. Substituting this expression back into eq.(4.9) gives

$$\Omega^{\eta_1 - \eta_2}(g; z_2) D^{i}_{1i'}(\mathcal{R}(g; z_2)) D^{j}_{2j'}(\mathcal{R}(g; z_2)) \mathcal{H}^{i'j'} = \mathcal{H}^{ij} \tag{4.13}$$

If $\eta_1 \neq \eta_2$ then for this equation to be true for any superconformal transformation, specially for superdilations, $\mathcal{H}^{i_1 i_2}$ should vanish. This completes our proof.
Specifically in the following, assuming that two superfields have the same scale dimension, η, we study the two-point correlation functions of scalar superfields and supervector fields.

4.1.1 Two-point Correlation Function of Scalar Superfield, $S(z)$

Obviously it has trivial representation and so the two-point function has the following unique form

$$\langle S(z_1) S(z_2) \rangle = \frac{c}{(x^2_{12} + \theta^2_{12}\bar{\theta}^2_{12})^{\eta}} \tag{4.14}$$

where c is constant. One example is Konishi current, $S(z) = \Phi(x_+, \theta)\bar{\Phi}(x_-, \bar{\theta})$, $\eta = 2$ in the interacting Wess-Zumino model. In free model we get $c = \frac{1}{4\pi^4}$.

4.1.2 Two-point Correlation Function of Supervector Field (Supercurrent), $V^{\mu}(z)$

We consider supervector fields, $V^{\mu}(z)$, of which the representation has the form

$$D^{\mu}{}_{\nu}(\mathcal{R}^{-1}(g; z)) = \pm_g \mathcal{R}^{-1\mu}{}_{\nu}(g; z) \tag{4.15}$$

and so $V^\mu(z)$ transforms under superconformal transformation as

$$V^\mu(z) \to \pm_g \Omega^\eta(g;z) \mathcal{R}^{-1\mu}{}_\nu(g;z) V^\nu(z') \tag{4.16}$$

Superconformal symmetric condition (4.6) is

$$\mathcal{H}^{\mu\nu} = L^\mu{}_\rho L^\nu{}_\lambda \mathcal{H}^{\rho\lambda} \tag{4.17}$$

Infinitesimally this becomes

$$\eta^{\mu\rho}\mathcal{H}^{\nu\lambda} - \eta^{\nu\rho}\mathcal{H}^{\mu\lambda} + \eta^{\mu\lambda}\mathcal{H}^{\rho\nu} - \eta^{\nu\lambda}\mathcal{H}^{\rho\mu} = 0 \tag{4.18}$$

Contracting with $\eta_{\nu\lambda}$ gives

$$\mathcal{H}^{\mu\rho} + 3\mathcal{H}^{\rho\mu} = \eta^{\mu\rho}\mathcal{H}^\nu{}_\nu = \mathcal{H}^{\rho\mu} + 3\mathcal{H}^{\mu\rho} \tag{4.19}$$

Hence $\mathcal{H}^{\mu\nu}$ is proportional to $\eta^{\mu\nu}$ and so the two-point function of supervector field, $V^\mu(z)$ with scale dimension, η has the following unique form

$$\langle V^\mu(z_1) V^\nu(z_2)\rangle = c\frac{I^{\mu\nu}(-z_{12})}{(x_{12}^2 + \theta_{12}^2\bar{\theta}_{12}^2)^\eta} \tag{4.20}$$

where c is constant. We confirmed this result by calculating the two-point functions of supercurrents in Wess-Zumino model and vector superfield theory respectively and checking them to coincide. We get $c = \frac{24}{\pi^4}$ in Wess-Zumino model and $c = \frac{1}{2\pi^4}$ in vector superfield theory. The conservation equation below is satisfied if and only if $\eta = 3$.

$$D^\alpha(z_1)\langle V_{\alpha\dot{\alpha}}(z_1) V_{\beta\dot{\beta}}(z_2)\rangle = 0 \tag{4.21}$$

where $V_{\alpha\dot{\alpha}} = \sigma_{\mu\alpha\dot{\alpha}} V^\mu$.

4.2 Three-Point Correlation Function of General Superfields

Here we will show that three-point function of superfields has the following general form

$$\langle \Psi_1^i(z_1)\Psi_2^j(z_2)\Psi_3^k(z_3)\rangle = \frac{D_{1i'}^i(I^{-1}(z_{13})) D_{2j'}^j(I^{-1}(z_{23}))\mathcal{H}^{i'j'k}(Z_3)}{(x_{12}^2 + \theta_{12}^2\bar{\theta}_{12}^2)^{\delta_3}(x_{23}^2 + \theta_{23}^2\bar{\theta}_{23}^2)^{\delta_1}(x_{13}^2 + \theta_{13}^2\bar{\theta}_{13}^2)^{\delta_2}} \tag{4.22}$$

where $\delta_1 = \frac{1}{2}(\eta_2 + \eta_3 - \eta_1)$, $\delta_2 = \frac{1}{2}(\eta_3 + \eta_1 - \eta_2)$, $\delta_3 = \frac{1}{2}(\eta_1 + \eta_2 - \eta_3)$ and

$$Z_3 = -i(z_{23}) \oplus i(z_{13}) \tag{4.23}$$

Explicitly $Z_3 = X_3,\ \Theta_3,\ \overline{\Theta}_3$ have the form

$$X_3 = \frac{x_{13}}{x_{13}^2 + \theta_{13}^2\bar{\theta}_{13}^2} - \frac{x_{23}}{x_{23}^2 + \theta_{23}^2\bar{\theta}_{23}^2} + i\frac{\theta_{23}x_{23+}\cdot\sigma\tilde{\sigma}x_{13-}\cdot\sigma\bar{\theta}_{13}}{x_{23+}^2 x_{13-}^2} - i\frac{\theta_{13}x_{13+}\cdot\sigma\tilde{\sigma}x_{23-}\cdot\sigma\bar{\theta}_{23}}{x_{13+}^2 x_{23-}^2}$$

$$\Theta_3 = i\frac{\tilde{\bar{\theta}}_{23}x_{23-}\cdot\tilde{\sigma}}{x_{23-}^2} - i\frac{\tilde{\bar{\theta}}_{13}x_{13-}\cdot\tilde{\sigma}}{x_{13-}^2}$$

$$\overline{\Theta}_3 = i\frac{x_{13+}\cdot\tilde{\sigma}\tilde{\theta}_{13}}{x_{13+}^2} - i\frac{x_{23+}\cdot\tilde{\sigma}\tilde{\theta}_{23}}{x_{23+}^2} \tag{4.24}$$

$\mathcal{H}^{ijk}(z)$ is of the form

$$\mathcal{H}^{ijk}(z) = \mathcal{H}_1^{ijk}(x) + \mathcal{H}^{ijk\mu}(x)\theta\sigma_\mu\bar{\theta} + \mathcal{H}_2^{ijk}(x)\frac{\theta^2\bar{\theta}^2}{x^2} \tag{4.25}$$

which satisfies

$$\mathcal{H}_a^{ijk}(rLx) = D^i_{1\,i'}(L)D^j_{2\,j'}(L)D^k_{3\,k'}(L)\mathcal{H}_a^{i'j'k'}(x) \tag{4.26}$$

$$\mathcal{H}^{ijk\mu}(rLx) = \pm_L\frac{1}{r}D^i_{1\,i'}(L)D^j_{2\,j'}(L)D^k_{3\,k'}(L)L^\mu{}_\nu\mathcal{H}^{i'j'k'\nu}(x) \tag{4.27}$$

where r, L are arbitrary real number & Lorentz transformation and $\pm_L = \det L/|\det L|$.
proof
Without loss of generality we can put

$$\langle\Psi_1^i(z_1)\Psi_2^j(z_2)\Psi_3^k(z_3)\rangle = \frac{D^i_{1\,i'}(I^{-1}(z_{13}))D^j_{2\,j'}(I^{-1}(z_{23}))H^{i'j'k}(i(z_{13}), i(z_{23}))}{(x_{12}^2 + \theta_{12}^2\bar{\theta}_{12}^2)^{\delta_3}(x_{23}^2 + \theta_{23}^2\bar{\theta}_{23}^2)^{\delta_1}(x_{13}^2 + \theta_{13}^2\bar{\theta}_{13}^2)^{\delta_2}} \tag{4.28}$$

With eq.(4.8), superconformal invariance of the correlation function leads to

$$H^{ijk}(i(z'_{13}), i(z'_{23})) = D^i_{1\,i'}(\mathcal{R}(g;z_3))D^j_{2\,j'}(\mathcal{R}(g;z_3))D^k_{3\,k'}(\mathcal{R}(g;z_3))H^{i'j'k'}(i(z_{13}), i(z_{23})) \tag{4.29}$$

We consider a superconformal transformation, $\tilde{g} : z \longrightarrow z'$ defined by

$$\tilde{g}(z) = z'_3 \oplus i(\tilde{z} \oplus i(-z_3 \oplus z)) \tag{4.30}$$

where $z'_3 = \tilde{g}(z_3)$, $\tilde{z}$ are arbitrary. $\tilde{z} \oplus i(z_{13}) = i(z'_{13})$ and $\mathcal{R}^\mu{}_\nu(\tilde{g};z_3) = \delta^\mu{}_\nu$ imply the supertranslational invariance of $H^{ijk}(z_1, z_2)$

$$H^{ijk}(\tilde{z} \oplus z_1, \tilde{z} \oplus z_2) = H^{ijk}(z_1, z_2) \tag{4.31}$$

Hence we can put

$$\mathcal{H}^{ijk}(z_{12}) = H^{ijk}(z_1, z_2) \tag{4.32}$$

which gives the general form of three-point correlation function (4.22). $\mathcal{H}^{ijk}(z)$ should satisfy the superconformal symmetric condition

$$\mathcal{H}^{ijk}(Z_3') = D^i_{1\,i'}(\mathcal{R}(g;z_3))D^j_{2\,j'}(\mathcal{R}(g;z_3))D^k_{3\,k'}(\mathcal{R}(g;z_3))\mathcal{H}^{i'j'k'}(Z_3) \tag{4.33}$$

where $g : z \to z'$ is an arbitrary superconformal transformation and $Z_3' = -i(z_{23}') \oplus i(z_{13}')$. However (nicely enough) $Z_3 = X_3, \Theta_3, \overline{\Theta}_3$ transforms to $Z_3' = X_3', \Theta_3', \overline{\Theta}_3'$ in a simple form. For any superconformal transformation

$$X_3'^{\mu} = \Omega^{-1}(g;z_3)\mathcal{R}^{\mu}{}_{\nu}(g;z_3)X_3^{\nu} \tag{4.34}$$

For continuous superconformal transformation

$$\Theta_3'^{\alpha} = \frac{\mathrm{sdet}_c^{1/3}}{\overline{\mathrm{sdet}}_c^{2/3}}\left.\frac{\partial^-\theta'^{\alpha}}{\partial\theta^{\beta}}\right|_{z_3}\Theta_3^{\beta}, \qquad \overline{\Theta}_3'^{\dot\alpha} = \frac{\overline{\mathrm{sdet}}_c^{1/3}}{\mathrm{sdet}_c^{2/3}}\left.\frac{\partial^+\bar\theta'^{\dot\alpha}}{\partial\bar\theta^{\dot\beta}}\right|_{z_3}\overline{\Theta}_3^{\dot\beta} \tag{4.35}$$

For superinversion-type transformation

$$\Theta_3'^{\alpha} = -\frac{\mathrm{sdet}_i^{1/3}}{\overline{\mathrm{sdet}}_i^{2/3}}\left.\frac{\partial^+\theta'^{\alpha}}{\partial\bar\theta^{\dot\alpha}}\right|_{z_3}\overline{\Theta}_3^{\dot\alpha}, \qquad \overline{\Theta}_3'^{\dot\alpha} = -\frac{\overline{\mathrm{sdet}}_i^{1/3}}{\mathrm{sdet}_i^{2/3}}\left.\frac{\partial^-\bar\theta'^{\dot\alpha}}{\partial\theta^{\alpha}}\right|_{z_3}\Theta_3^{\alpha} \tag{4.36}$$

and so $\Theta_3\sigma^{\mu}\overline{\Theta}_3$ transforms like a pseudo vector at z_3 under superconformal transformation

$$\Theta_3'\sigma^{\mu}\overline{\Theta}_3' = \pm_g\Omega^{-1}(g;z_3)\mathcal{R}^{\mu}{}_{\nu}(g;z_3)\Theta_3\sigma^{\mu}\overline{\Theta}_3 \tag{4.37}$$

To verify these, we only need to check for the fundamental elements of superconformal group, since superdeterminant and $\frac{\partial^-\theta'}{\partial\theta}, \frac{\partial^+\bar\theta'}{\partial\bar\theta}, \frac{\partial^+\theta'}{\partial\bar\theta}, \frac{\partial^-\bar\theta'}{\partial\theta}$ are all representations of superconformal group. Supertranslations, superdilations and super Lorentz transformation cases are straightforward. For superinversion we consider a superconformal transformation, $f : z \to z'$ defined by

$$f(z) = i(-i(z_3)) \oplus i(z_3 \oplus i(z)) \tag{4.38}$$

Z_3' and Z_3 are related by $Z_3' = -f(i(z_{23})) \oplus f(i(z_{23} \oplus Z_3))$ and from eq.(A.4) we get

$$x_-' \cdot \sigma = (z_3 \oplus i(z))_-^{\mu}\sigma_{\mu}\ x_+ \cdot \tilde\sigma\ x_{3+} \cdot \sigma \tag{4.39}$$

$$i\bar\theta_{\dot\alpha}' = (i^{\alpha}(z_3 \oplus i(z)) - i^{\alpha}(z_3))x_-' \cdot \sigma_{\alpha\dot\alpha} \tag{4.40}$$

Now direct calculation leads to

$$f(z)=i(-i(z_3))\oplus((x_3^2+\theta_3^2\bar{\theta}_3^2)I^{\mu}_{s\nu}(z_3)x^{\nu},i\frac{x_{3+}^2}{x_{3-}^2}(\tilde{\bar{\theta}}x_{3-}\cdot\tilde{\sigma})^{\alpha},-i\frac{x_{3-}^2}{x_{3+}^2}(x_{3+}\cdot\tilde{\sigma}\tilde{\theta})^{\dot{\alpha}}) \tag{4.41}$$

which verifies eqs.(4.34,4.36) for superinversion.
Invariance under superdilations restricts the power series expansion of $\mathcal{H}^{ijk}(z)$ in $\theta,\bar{\theta}$ as

$$\mathcal{H}^{ijk}(z)=\mathcal{H}_1^{ijk}(x)+\mathcal{H}^{ijk\mu}(x)\theta\sigma_{\mu}\bar{\theta}+\mathcal{H}_2^{ijk}(x)\frac{\theta^2\bar{\theta}^2}{x^2} \tag{4.42}$$

Therefore $\mathcal{H}^{ijk}(z)$ is a function of $x,\theta\sigma\bar{\theta}$ (or $x_{\pm}$) and the superconformal invariance of $\mathcal{H}^{ijk}$ (4.33) is equivalent to

$$\mathcal{H}^{ijk}(rLx,\pm_L rL\theta\sigma\bar{\theta})=D_{1\,i'}^{i}(L)D_{2\,j'}^{j}(L)D_{3\,k'}^{k}(L)\mathcal{H}^{i'j'k'}(x,\theta\sigma\bar{\theta}) \tag{4.43}$$

where r and L are arbitrary real number and Lorentz transformation. This completes our proof.
Our expression for the three-point function (4.22) is asymmetric in its treatment of superfields. However if we define

$$Z_1=-i(z_{31})\oplus i(z_{21})\qquad Z_2=-i(z_{12})\oplus i(z_{32}) \tag{4.44}$$

then using eqs.(A.3,A.4,A.5) one can get

$$X^{\mu}_{3\pm}=\frac{x_{12}^2+\theta_{12}^2\bar{\theta}_{12}^2}{x_{23}^2+\theta_{23}^2\bar{\theta}_{23}^2}(I(z_{13})I(Z_1))^{\mu}{}_{\nu}X^{\nu}_{1\pm}=\frac{x_{12}^2+\theta_{12}^2\bar{\theta}_{12}^2}{x_{13}^2+\theta_{13}^2\bar{\theta}_{13}^2}(I(z_{23})I(Z_2))^{\mu}{}_{\nu}X^{\nu}_{2\pm} \tag{4.45}$$

$$\begin{aligned}I^{-1}(z_{31})I(Z_1)I(z_{21})&=I(z_{23})\\ I^{-1}(z_{12})I(Z_2)I(z_{32})&=I(z_{31})\\ I^{-1}(z_{23})I(Z_3)I(z_{13})&=I(z_{12})\end{aligned} \tag{4.46}$$

Now by virtue of superconformal symmetry of $\mathcal{H}^{ijk}$ one can recover a democratic way of treating superfields as

$$\begin{aligned}&D_{1i'}^{i}(I^{-1}(z_{13}))D_{2j'}^{j}(I^{-1}(z_{23}))\mathcal{H}^{i'j'k}(X_{3\pm})\\ =\;&D_{2j'}^{j}(I^{-1}(z_{21}))D_{3k'}^{k}(I^{-1}(z_{31}))\mathcal{F}^{ij'k'}(X_{1\pm})\\ =\;&D_{1i'}^{i}(I^{-1}(z_{12}))D_{3k'}^{k}(I^{-1}(z_{32}))\mathcal{G}^{i'jk'}(X_{2\pm})\end{aligned} \tag{4.47}$$

where

$$\mathcal{F}^{ijk}(X_{1\pm}) = D^{i}_{1i'}(I(Z_1)D^{k}_{3k'}(I(Z_1))\mathcal{H}^{i'jk'}(X_{1\pm})$$
$$\mathcal{G}^{ijk}(X_{2\pm}) = D^{i}_{1i'}(I(Z_2))\mathcal{H}^{i'jk}(I(Z_2)X_{2\mp}) \tag{4.48}$$

Specially for bosonic superfields belonging to a same type, there are additional conditions on $\mathcal{H}$ due to the invariance of Green function under permutations of superfields.

- for $1 \leftrightarrow 2$

$$\mathcal{H}^{ijk}(x, \theta\sigma\bar{\theta}) = \mathcal{H}^{jik}(-x, \theta\sigma\bar{\theta}) \tag{4.49}$$

- for $2 \leftrightarrow 3$

$$D^{i}{}_{i'}(I(z))\mathcal{H}^{i'jk}(x, \theta\sigma\bar{\theta}) = \mathcal{H}^{ikj}(x, -I(z)\theta\sigma\bar{\theta}) \tag{4.50}$$

4.2.1 Three-point Correlation Function of Scalar Superfield, $S(z)$

$\mathcal{H}(z)$ has the form

$$\mathcal{H}(z) = \mathcal{H}_1(x) + \mathcal{H}^{\mu}(x)\theta\sigma_{\mu}\bar{\theta} + \mathcal{H}_2(x)\frac{\theta^2\bar{\theta}^2}{x^2} \tag{4.51}$$

Superconformal invariance of $\mathcal{H}(z)$ is

$$\mathcal{H}_a(rLx) = \mathcal{H}_a(x), \qquad \mathcal{H}^{\mu}(rLx) = \pm_L \frac{1}{r} L^{\mu}{}_{\nu}\mathcal{H}^{\nu}(x) \tag{4.52}$$

where r, L are arbitrary real number and Lorentz transformation. This implies that $\mathcal{H}_1(x)$, $\mathcal{H}_2(x)$ are constant and $\mathcal{H}^{\mu}(x)$ is linear in x^{μ}/x^2 , but under superinversion it should also change the sign, therefore $\mathcal{H}^{\mu}(x) = 0$ and so three-point function of scalar superfield has the following general form

$$\langle S(z_1)S(z_2)S(z_3)\rangle = \frac{c_1 + c_2\dfrac{\Theta_3^2\overline{\Theta}_3^2}{X_3^2}}{(x_{12}^2 + \theta_{12}^2\bar{\theta}_{12}^2)^{\eta/2}(x_{23}^2 + \theta_{23}^2\bar{\theta}_{23}^2)^{\eta/2}(x_{13}^2 + \theta_{13}^2\bar{\theta}_{13}^2)^{\eta/2}} \tag{4.53}$$

Although this expression looks asymmetric, one can show that $\frac{\Theta_3^2\overline{\Theta}_3^2}{X_3^2}$ is a symmetric quantity under permutations of z_1, z_2, z_3.
proof
From

$$x_{12-}\cdot\sigma = x_{23+}\cdot\sigma X_{3+}\cdot\sigma x_{13-}\cdot\sigma \tag{4.54}$$

we get

$$X_{3+}^2 = \frac{x_{21+}^2}{x_{23+}^2 x_{31+}^2} \qquad X_{3-}^2 = \frac{x_{21-}^2}{x_{23-}^2 x_{31-}^2} \tag{4.55}$$

hence

$$\begin{aligned} X_3^2 + \Theta_3^2\overline{\Theta}_3^2 &= \frac{x_{12}^2 + \theta_{12}^2\bar{\theta}_{12}^2}{(x_{23}^2 + \theta_{23}^2\bar{\theta}_{23}^2)(x_{31}^2 + \theta_{31}^2\bar{\theta}_{31}^2)} \\ X_3^2 + 2\Theta_3^2\overline{\Theta}_3^2 &= \tfrac{1}{2}(x_{12}^2 + \theta_{12}^2\bar{\theta}_{12}^2)^2\left(\tfrac{1}{x_{12+}^2 x_{23+}^2 x_{31+}^2} + \tfrac{1}{x_{12-}^2 x_{23-}^2 x_{31-}^2}\right) \end{aligned} \tag{4.56}$$

Combining these two shows

$$\begin{aligned} \frac{\Theta_3^2\overline{\Theta}_3^2}{X_3^2} &= \tfrac{1}{2}(x_{12}^2 + \theta_{12}^2\bar{\theta}_{12}^2)(x_{23}^2 + \theta_{23}^2\bar{\theta}_{23}^2)(x_{31}^2 + \theta_{31}^2\bar{\theta}_{31}^2)\left(\tfrac{1}{x_{12+}^2 x_{23+}^2 x_{31+}^2} + \tfrac{1}{x_{12-}^2 x_{23-}^2 x_{31-}^2}\right) - 1 \\ &= \frac{\theta_{12}^2\bar{\theta}_{12}^2}{x_{12}^2} + \frac{\theta_{23}^2\bar{\theta}_{23}^2}{x_{23}^2} + \frac{\theta_{31}^2\bar{\theta}_{31}^2}{x_{31}^2} - 4\frac{\theta_{12}x_{12}\cdot\sigma\bar{\theta}_{12}\theta_{23}x_{23}\cdot\sigma\bar{\theta}_{23}}{x_{12}^2 x_{23}^2} \\ &\quad - 4\frac{\theta_{23}x_{23}\cdot\sigma\bar{\theta}_{23}\theta_{31}x_{31}\cdot\sigma\bar{\theta}_{31}}{x_{23}^2 x_{31}^2} - 4\frac{\theta_{31}x_{31}\cdot\sigma\bar{\theta}_{31}\theta_{12}x_{12}\cdot\sigma\bar{\theta}_{12}}{x_{31}^2 x_{12}^2} \\ &= \frac{\Theta_1^2\overline{\Theta}_1^2}{X_1^2} = \frac{\Theta_2^2\overline{\Theta}_2^2}{X_2^2} \end{aligned} \tag{4.57}$$

This completes our proof.
For Konishi current, $S(z) = \Phi(x_+,\theta)\bar{\Phi}(x_-,\bar{\theta}),\ \eta = 2$, in massless free Wess-Zumino model, by direct calculation we confirmed this result with $c_1 = c_2 = \frac{1}{4\pi^6}$.

4.2.2 Three-point Correlation Function of Supercurrent, $V^\mu(z)$

$\mathcal{H}^{\lambda\mu\nu}(z)$ has the form

$$\mathcal{H}^{\lambda\mu\nu}(z) = \mathcal{H}_1^{\lambda\mu\nu}(x) + \mathcal{H}^{\lambda\mu\nu\kappa}(x)\theta\sigma_\kappa\bar{\theta} + \mathcal{H}_2^{\lambda\mu\nu}(x)\frac{\theta^2\bar{\theta}^2}{x^2} \tag{4.58}$$

Superconformal invariance of $\mathcal{H}^{\lambda\mu\nu}(z)$ is

$$\begin{aligned} \mathcal{H}_a^{\lambda\mu\nu}(rLx) &= \pm_L L^\lambda{}_{\lambda'} L^\mu{}_{\mu'} L^\nu{}_{\nu'} \mathcal{H}_a^{\lambda'\mu'\nu'}(x) \\ \mathcal{H}^{\lambda\mu\nu\kappa}(rLx) &= \frac{1}{r} L^\lambda{}_{\lambda'} L^\mu{}_{\mu'} L^\nu{}_{\nu'} L^\kappa{}_{\kappa'} \mathcal{H}^{\lambda'\mu'\nu'\kappa'}(x) \end{aligned} \tag{4.59}$$

and so the most general form of $\mathcal{H}^{\lambda\mu\nu}(z)$ is

$$\begin{aligned}
\mathcal{H}^{\lambda\mu\nu}(z) = \quad & c_1\epsilon^{\lambda\mu\nu\kappa}\frac{x_\kappa}{(x^2)^{\frac{1}{2}}} \\
& +h_1\frac{x^\lambda x^\mu x^\nu \theta x\cdot\sigma\bar{\theta}}{(x^2)^{\frac{5}{2}}} + h_2\frac{x^\lambda x^\mu \theta\sigma^\nu\bar{\theta}}{(x^2)^{\frac{3}{2}}} + h_3\frac{x^\mu x^\nu \theta\sigma^\lambda\bar{\theta}}{(x^2)^{\frac{3}{2}}} + h_4\frac{x^\nu x^\lambda \theta\sigma^\mu\bar{\theta}}{(x^2)^{\frac{3}{2}}} \\
& +(h_5 x^\lambda\eta^{\mu\nu} + h_6 x^\mu\eta^{\nu\lambda} + h_7 x^\nu\eta^{\lambda\mu})\frac{\theta x\cdot\sigma\bar{\theta}}{(x^2)^{\frac{3}{2}}} \\
& +(h_8\eta^{\lambda\mu}\eta^{\nu\kappa} + h_9\eta^{\mu\nu}\eta^{\lambda\kappa} + h_{10}\eta^{\nu\lambda}\eta^{\mu\kappa})\frac{\theta\sigma_\kappa\bar{\theta}}{(x^2)^{\frac{1}{2}}} \\
& +c_2\epsilon^{\lambda\mu\nu\kappa}x_\kappa\frac{\theta^2\bar{\theta}^2}{(x^2)^{\frac{3}{2}}}
\end{aligned} \tag{4.60}$$

which can be obtained in the following way: For each point in spacetime, x, by taking a Lorentz transformation and rescaling it properly, one can transform it to a unit vector. By considering the infinitesimal action of the *little* Lorentz group of it as eq.(4.59), which leaves the unit vector invariant, one can get the general forms of $H_a^{\kappa\lambda\mu}$, $H^{\kappa\lambda\mu\nu}$ at the unit vector. Then transforming the unit vector back to the original vector, x, gives the general solutions of eq.(4.59) when L is continuous Lorentz transformation. To get the final form (4.60) one only needs to impose the extra condition, invariance under superinversion, on them.
For supervector fields belonging to a same type there are two additional restrictions

$$\begin{aligned}
&\mathcal{H}^{\lambda\mu\nu}(x,\theta\sigma\bar{\theta}) = \mathcal{H}^{\mu\lambda\nu}(-x,\theta\sigma\bar{\theta}) \\
&I^\kappa{}_\lambda(z)\mathcal{H}^{\lambda\mu\nu}(x,\theta\sigma\bar{\theta}) = -\mathcal{H}^{\kappa\nu\mu}(x,-I(z)\theta\sigma\bar{\theta})
\end{aligned} \tag{4.61}$$

these two restrictions imply

$$\begin{aligned}
h_3 &= h_4 \qquad & h_2 + h_3 + 2h_8 + 2c_1 &= 0 \\
h_5 &= h_6 \qquad & h_6 + h_7 + 2h_8 + 2c_1 &= 0 \\
h_9 &= h_{10} \qquad & h_9 &= h_8 + 2c_1
\end{aligned} \tag{4.62}$$

Thus, there exist six independent parameters, including one for overall constant, in the three-point function of a supervector field. However, if the three-point function satisfies the

conservation equation

$$0 = \frac{\partial}{\partial x_1^\lambda} \langle V^\lambda(z_1) V^\mu(z_2) V^\nu(z_3) \rangle \tag{4.63}$$

then one can show that the scale dimension, η should be 3 and the three-point function has the following general form with two free parameters, c, d

$$\begin{aligned} \langle V^\lambda(z_1) V^\mu(z_2) V^\nu(z_3) \rangle = & \frac{I^\lambda{}_\kappa(-z_{13})}{(x_{13}^2 + \theta_{13}^2 \bar\theta_{13}^2)^3} \frac{I^\mu{}_\rho(-z_{23})}{(x_{23}^2 + \theta_{23}^2 \bar\theta_{23}^2)^3} \\ & \times \left(c\mathcal{J}^{\kappa\rho\nu}(Z_3) + d\mathcal{K}^{\kappa\rho\nu}(Z_3) \right) \end{aligned} \tag{4.64}$$

where

$$Z_3 = -i(z_{23}) \oplus i(z_{13}) \tag{4.65}$$

$$\begin{aligned} \mathcal{J}^{\mu\nu\rho}(z) = \epsilon^{\mu\nu\rho\kappa} \frac{x_\kappa}{(x^2)^2} - 4 \frac{x_\kappa x_\lambda \theta\sigma_\tau\bar\theta}{(x^2)^3} (\eta^{\mu\rho}\mathcal{E}^{\kappa\lambda,\tau\nu} + \eta^{\nu\rho}\mathcal{E}^{\kappa\lambda,\tau\mu} - \eta^{\mu\nu}\mathcal{E}^{\kappa\lambda,\tau\rho}) \\ \mathcal{E}^{\kappa\lambda,\mu\nu} = \tfrac{1}{2}(\eta^{\kappa\mu}\eta^{\lambda\nu} + \eta^{\kappa\nu}\eta^{\lambda\mu}) - \tfrac{1}{4}\eta^{\kappa\lambda}\eta^{\mu\nu} \end{aligned} \tag{4.66}$$

$$\begin{aligned} \mathcal{K}^{\mu\nu\rho}(z) = & \frac{\theta\sigma_\kappa\bar\theta}{(x^2)^4} \Big((x^2)^2 (\eta^{\mu\nu}\eta^{\rho\kappa} + \eta^{\kappa\mu}\eta^{\nu\rho} + \eta^{\rho\mu}\eta^{\nu\kappa}) \\ & + 4x^2 x^\rho (x^\mu \eta^{\nu\kappa} + x^\nu \eta^{\kappa\mu} + x^\kappa \eta^{\mu\nu}) \\ & - 6x^2 (x^\mu x^\nu \eta^{\rho\kappa} + x^\mu x^\kappa \eta^{\rho\nu} + x^\nu x^\kappa \eta^{\rho\mu}) - 12 x^\mu x^\nu x^\rho x^\kappa \Big) \end{aligned} \tag{4.67}$$

$\mathcal{E}^{\kappa\lambda,\mu\nu}$ is the four dimensional projection operator, which transforms any 4×4 matrix to traceless and symmetric one. Furthermore, eq.(4.64) satisfies the "strong" conservation equation

$$0 = D^\alpha(z_1) \langle V_{\alpha\dot\alpha}(z_1) V_{\beta\dot\beta}(z_2) V_{\gamma\dot\gamma}(z_3) \rangle \tag{4.68}$$

although one might guess that eq.(4.63) is a weaker condition than eq.(4.68) due to the commutation relation $\{D_\alpha, \overline{D}_{\dot\alpha}\} = -2i\partial_{\alpha\dot\alpha}$. This result is demonstrated in appendix B.

5 Summary & Discussion

Supertranslations, superdilations, super Lorentz transformations and superinversion are all the fundamental elements of the $N = 1$ four dimensional superconformal group, in the sense that they generate all the superconformal transformations. There are 4×4 and

2×2 representations of superconformal group. Under superconformal transformations the left invariant derivatives and some class of superfields (chiral/anti-chiral superfields, supercurrents in Wess-Zumino model and vector superfield theory) follow these representations without auxiliary terms appearing, and so they are quasi-primary. Due to the superconformal symmetric property, the two-point correlation function of supercurrents is unique up to a overall constant and the general form of the three-point function of supercurrents has two free parameters. Even if the supercurrent is not conserved as in the interacting theories, the two-point function of supercurrents is unique. Readers may refer our result that two-point function of superfields with different scale dimensions vanishes and acting derivatives on superfields changes the scale dimension, although generally the descendent fields, differentiated quasi-primary fields, do not have transformation rules in close forms. On the other hand, we expect that the general form of the three-point correlation functions of supercurrents in the interacting theories may have at most six independent parameters, including an overall constant but this number can be reduced by considering the general forms of the correlation functions of supercurrent, J^μ and its divergence, $\partial_\mu J^\mu$. It would be of interest to get operator product expansions of superfields using our results. It is straightforward to obtain the infinitesimal superconformal transformation rules for superfields and so Ward identities from our results using such as $\delta \mathrm{sdet} M = \delta \mathrm{str} M$.

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Appendix

A Superconformal Invariance of $I(z)$

Here we derive eq.(4.8)

$$\mathcal{R}^{-1}(g; z_2) I(z'_{12}) \mathcal{R}(g; z_1) = I(z_{12}) \tag{A.1}$$

where $z_{12} = -z_2 \oplus z_1, z'_{12} = -z'_2 \oplus z'_1$ and $g : z \to z'$ is an arbitrary superconformal transformation.

proof

It is straightforward to verify this for supertranslations, super Lorentz transformations and superdilations. Superinversion case is not simple as usual. Direct calculation using

$$\sigma^{\mu}\tilde{\sigma}^{\lambda}\sigma^{\nu} = \eta^{\mu\nu}\sigma^{\lambda} - \eta^{\lambda\nu}\sigma^{\mu} - \eta^{\lambda\mu}\sigma^{\nu} + i\epsilon^{\mu\lambda\nu\rho}\sigma_{\rho} \tag{A.2}$$

gives

$$\begin{aligned} x_{+}\cdot\sigma\ \tilde{\sigma}^{\mu}\ x_{-}\cdot\sigma &= (x^2+\theta^2\bar{\theta}^2)I^{\mu}{}_{\nu}(z)\sigma^{\nu} \\ x_{-}\cdot\sigma\ \tilde{\sigma}^{\mu}\ x_{+}\cdot\sigma &= (x^2+\theta^2\bar{\theta}^2)I^{-1\mu}{}_{\nu}(z)\sigma^{\nu} \end{aligned} \tag{A.3}$$

and under superinversion one can show

$$\begin{aligned} x_{1+}\cdot\sigma\ x'_{12-}\cdot\tilde{\sigma}\ x_{2-}\cdot\sigma &= x_{12+}\cdot\sigma \\ x_{2+}\cdot\sigma\ x'_{12+}\cdot\tilde{\sigma}\ x_{1-}\cdot\sigma &= x_{12-}\cdot\sigma \end{aligned} \tag{A.4}$$

These two relations (A.3,A.4) imply

$$I^{-1}(z_2)I(z'_{12})I(z_1) = I(z_{12}) \tag{A.5}$$

Hence eq.(4.8) holds for each fundamental element of superconformal group and also under the successive superconformal transformations $z \xrightarrow{g} z' \xrightarrow{g'} z''$ this equation is preserved.

$$\mathcal{R}^{-1}(g'\circ g; z_2)I(z''_{12})\mathcal{R}(g'\circ g; z_1) = I(z_{12}) \tag{A.6}$$

Thus, eq.(A.1) holds for any superconformal transformation.

B Three-point Correlation Function of Supercurrent

Here we show first that the three-point function of supervector fields satisfying the strong (superficially) conservation equation (4.68) has the form (4.64) and that the weak (superficially) conservation equation (4.63) is actually equivalent to the strong one.
Using eq.(4.56) we write the three-point function as

$$\langle V^{\mu}(z_1)V^{\nu}(z_2)V^{\rho}(z_3)\rangle = \frac{I^{\mu}{}_{\kappa}(-z_{13})I^{\nu}{}_{\lambda}(-z_{23})\mathcal{H}^{\kappa\lambda\nu}(Z_3)}{(x_{13}^2+\theta_{13}^2\bar{\theta}_{13}^2)^{\eta}(x_{23}^2+\theta_{23}^2\bar{\theta}_{23}^2)^{\eta}(X_3^2+\Theta_3^2\overline{\Theta}_3^2)^{\eta/2}} \tag{B.1}$$

With $V_{\alpha\dot\alpha} = \sigma_{\mu\alpha\dot\alpha}V^\mu$ and $D_\alpha(z_1) = D_\alpha(z_{13}) = -i\frac{x_{13+}\cdot\sigma_{\alpha\dot\alpha}}{x_{13-}^2}\overline{D}^{\dot\alpha}(Z_3)$, the "strong" conservation equation (4.68) is equivalent to

$$\begin{aligned}
0 = \quad & D^\alpha(z_1)\left(\frac{(x_{13+}\cdot\sigma\tilde\sigma_\mu x_{13-}\cdot\sigma)_{\alpha\dot\alpha}}{(x_{13+}^2x_{13-}^2)^{\frac{\eta+1}{2}}}\frac{\mathcal{H}^{\mu\nu\lambda}(Z_3)}{(X_3^2+\Theta_3^2\overline{\Theta}_3^2)^{\frac{\eta}{2}}}\right) \\
= \quad & 2i(3-\eta)\frac{(\epsilon x_{13-}\cdot\tilde\sigma\sigma_\mu\bar\theta_{13})_{\dot\alpha}}{(x_{13}^2+\theta_{13}^2\bar\theta_{13}^2)^{\eta+1}}\frac{\mathcal{H}^{\mu\nu\lambda}(Z_3)}{(X_3^2+\Theta_3^2\overline{\Theta}_3^2)^{\frac{\eta}{2}}} \\
& -i\frac{(x_{13+}^2)^2}{(x_{13+}^2x_{13-}^2)^{\frac{\eta+3}{2}}}(\tilde\sigma_\mu x_{13-}\cdot\sigma)^{\dot\beta}{}_{\dot\alpha}\overline{D}_{\dot\beta}(Z_3)\frac{\mathcal{H}^{\mu\nu\lambda}(Z_3)}{(X_3^2+\Theta_3^2\overline{\Theta}_3^2)^{\frac{\eta}{2}}}
\end{aligned} \tag{B.2}$$

Multiplying $(x_{13+}^2)^{\frac{\eta-1}{2}}(x_{13-}^2)^{\frac{\eta+1}{2}}x_{13-}\cdot\tilde\sigma^{\dot\alpha\alpha}$ gives

$$2(3-\eta)\frac{x_{13-}^2}{x_{13+}^2}(\epsilon\sigma_\mu\bar\theta_{13})^\alpha\frac{\mathcal{H}^{\mu\nu\lambda}(Z_3)}{(X_3^2+\Theta_3^2\overline{\Theta}_3^2)^{\frac{\eta}{2}}} = \tilde\sigma_\mu^{\dot\alpha\alpha}\overline{D}_{\dot\alpha}(Z_3)\frac{\mathcal{H}^{\mu\nu\lambda}(Z_3)}{(X_3^2+\Theta_3^2\overline{\Theta}_3^2)^{\frac{\eta}{2}}} \tag{B.3}$$

Left hand side is a function of z_{13}, Z_3 and right hand side is a function of Z_3 only. For this identity to hold always, both sides should vanish. Hence $\eta = 3$ and

$$0 = \tilde\sigma_\mu^{\dot\alpha\alpha}(\bar\partial_{\dot\alpha} + i(\theta\sigma^\rho)_{\dot\alpha}\partial_\rho)\frac{\mathcal{H}^{\mu\nu\lambda}(z)}{(x^2+\theta^2\bar\theta^2)^{3/2}} \tag{B.4}$$

Explicitly

$$0 = \tilde\sigma_\mu^{\dot\alpha\alpha}(\bar\partial_{\dot\alpha} + i(\theta\sigma^\rho)_{\dot\alpha}\partial_\rho)\left(\begin{array}{l}
c_1\epsilon^{\mu\nu\lambda\kappa}\dfrac{x_\kappa}{(x^2)^2} \\
+\Big(k_1x^\mu x^\nu x^\lambda x^\kappa + k_2x^2x^\mu x^\nu\eta^{\lambda\kappa} + k_3x^2x^\lambda(x^\mu\eta^{\nu\kappa}+x^\nu\eta^{\mu\kappa}) \\
\quad + k_4x^2x^\kappa(x^\mu\eta^{\nu\lambda}+x^\nu\eta^{\mu\lambda}) + k_5x^2x^\kappa x^\lambda\eta^{\mu\nu} + k_6(x^2)^2\eta^{\mu\nu}\eta^{\lambda\kappa} \\
\quad + k_7(x^2)^2(\eta^{\mu\kappa}\eta^{\nu\lambda}+\eta^{\mu\lambda}\eta^{\nu\kappa})\Big)\dfrac{\theta\sigma_\kappa\bar\theta}{(x^2)^4} \\
+(c_2-\frac{3}{2}c_1)\epsilon^{\mu\nu\lambda\kappa}x_\kappa\dfrac{\theta^2\bar\theta^2}{(x^2)^3}
\end{array}\right) \tag{B.5}$$

where

$$2c_1 = k_2 + k_3 + 2k_7 \qquad 2c_1 = k_4 + k_5 + 2k_7 \qquad 2c_1 = k_7 - k_6 \tag{B.6}$$

Eq.(B.5) has two sorts of terms, θ-term and $\theta^2\bar{\theta}$-term, both of which are required to be zero.

For θ-term.
θ-term is of the form $(\theta\sigma^\mu\tilde{\sigma}^\nu)^\alpha K_{\mu\nu}(z) = 0$. Using $\sigma^\mu_{\alpha\dot{\alpha}}\tilde{\sigma}^{\dot{\beta}\beta}_\mu = -2\delta^\beta_\alpha\delta^{\dot{\beta}}_{\dot{\alpha}}$ one can easily show that $0 = \sigma^\mu\tilde{\sigma}^\nu K_{\mu\nu}(z)$ is equivalent to

$$0 = \mathrm{Tr}(\sigma^\kappa\tilde{\sigma}^\lambda\sigma^\mu\tilde{\sigma}^\nu)K_{\mu\nu}(z) = 2(\eta^{\kappa\lambda}\eta^{\mu\nu} + \eta^{\kappa\nu}\eta^{\lambda\mu} - \eta^{\kappa\mu}\eta^{\lambda\nu} - i\epsilon^{\kappa\lambda\mu\nu})K_{\mu\nu}(z) \tag{B.7}$$

Hence we have

$$\begin{aligned} 0 = \quad & (\eta^{\varsigma\tau}\delta^\kappa_\mu + \eta^{\tau\kappa}\delta^\varsigma_\mu - \eta^{\varsigma\kappa}\delta^\tau_\nu - i\epsilon^{\varsigma\tau\kappa}{}_\mu)\Big\{-ic_1\epsilon^{\mu\nu\lambda\rho}((x^2)^2\eta_{\rho\kappa} - 4x^2x_\rho x_\kappa) + k_1x_\kappa x^\mu x^\nu x^\lambda \\ & + k_2x^2x^\mu x^\nu\delta^\lambda_\kappa + k_3x^2x^\lambda(x^\mu\delta^\nu_\kappa + x^\nu\delta^\mu_\kappa) + k_4x^2x_\kappa(x^\mu\eta^{\nu\lambda} + x^\nu\eta^{\mu\lambda}) + k_5x_\kappa x^\lambda\eta^{\mu\nu}x^2 \\ & + k_6\eta^{\mu\nu}\delta^\lambda_\kappa(x^2)^2 + k_7(\eta^{\nu\lambda}\delta^\mu_\kappa + \eta^{\mu\lambda}\delta^\nu_\kappa)\Big\} \end{aligned} \tag{B.8}$$

Symmetric part for ς, τ leads to

$$0 = (k_1 + k_2 + 5k_3 + k_4 + k_5)x^\nu x^\lambda + (k_4 + k_6 + 5k_7)\eta^{\nu\lambda}x^2 \tag{B.9}$$

and so

$$k_1 + k_2 + 5k_3 + k_4 + k_5 = 0 \qquad k_4 + k_6 + 5k_7 = 0 \tag{B.10}$$

By $\epsilon^{\varsigma\tau}{}_{\kappa\mu}\epsilon^{\kappa\mu\lambda\nu} = 2(\eta^{\varsigma\nu}\eta^{\tau\lambda} - \eta^{\varsigma\lambda}\eta^{\tau\nu})$, anti-symmetric part leads to

$$\begin{aligned} 0 = \quad & i\Big\{4c_1\epsilon^{\varsigma\tau\nu\lambda}x^2 - (4c_1\epsilon^{\rho\tau\nu\lambda}x^\varsigma + 4c_1\epsilon^{\varsigma\rho\nu\lambda}x^\tau + (k_2 - k_4)\epsilon^{\varsigma\tau\rho\lambda}x^\nu + (k_5 - k_3)\epsilon^{\varsigma\tau\nu\rho}x^\lambda)x_\rho\Big\} \\ & +(4c_1 - k_2 + k_4)x^\varsigma x^\nu\eta^{\lambda\tau} + (4c_1 + k_3 - k_5)x^\tau x^\lambda\eta^{\varsigma\nu} - (4c_1 + k_3 - k_5)x^\varsigma x^\lambda\eta^{\tau\nu} \\ & -(4c_1 - k_2 + k_4)x^\tau x^\nu\eta^{\varsigma\lambda} \end{aligned} \tag{B.11}$$

Considering the case when $\varsigma, \tau, \nu, \lambda$ are all different gives

$$0 = 4c_1 - k_2 + k_4 \qquad 0 = 4c_1 + k_3 - k_5 \tag{B.12}$$

and so eq.(B.11) becomes

$$c_1\{(\epsilon^{\rho\tau\nu\lambda}x^\varsigma + \epsilon^{\varsigma\rho\nu\lambda}x^\tau + \epsilon^{\varsigma\tau\rho\lambda}x^\nu + \epsilon^{\varsigma\tau\nu\rho}x^\lambda)x_\rho - \epsilon^{\varsigma\tau\nu\lambda}x^2\} = 0 \tag{B.13}$$

which holds automatically regardless of the value of c_1.

For $\theta^2\bar{\theta}$-term.
Direct calculation gives

$$
\begin{aligned}
0 = \quad & (4k_1 + 12k_3)x^\nu x^\lambda x \cdot \sigma \\
& -(k_1 + k_2 + 3k_3 + k_4 + k_5 + 4k_6 + 4k_7)(x^\nu \sigma^\lambda + x^\lambda \sigma^\nu)x^2 \\
& +(k_2 + k_3 + 3k_4 + k_5 + 4k_6 + 12k_7)\eta^{\nu\lambda} x \cdot \sigma x^2 \\
& +i(4c_2 - 6c_1 - k_2 + k_3 + k_4 - k_5 - 4k_6 + 4k_7)\epsilon^{\nu\lambda\kappa\rho} x_\kappa \sigma_\rho x^2
\end{aligned}
\tag{B.14}
$$

Various choices of $\nu, \lambda, x^\nu, x^\lambda$ can confirm that each term should vanish, hence

$$
\begin{aligned}
0 &= k_1 + 3k_3 \\
0 &= k_2 + k_4 + k_5 + 4k_6 + 4k_7 \\
0 &= k_2 + k_3 + 3k_4 + k_5 + 4k_6 + 12k_7 \\
0 &= 4c_2 - 6c_1 - k_2 + k_3 + k_4 - k_5 - 4k_6 + 4k_7
\end{aligned}
\tag{B.15}
$$

the solution of eqs.(B.6,B.10,B.12,B.15) may be written in terms of two parameters, c, d

$$
\begin{array}{lllll}
c_1 = c & c_2 = \frac{3}{2}c & k_1 = -12d & k_2 = -6d & k_3 = 4d \\
k_4 = -4c - 6d & k_5 = 4c + 4d & k_6 = -c + d & k_7 = c + d &
\end{array}
\tag{B.16}
$$

which determines the three-point function of conserved supervector fields as eq.(4.64). Now we will show that the "weak" conservation condition (4.63) actually implies the "strong" one (4.68). By the transformation rule for ∂_μ (2.51), the "weak" conservation condition is equivalent to

$$
\begin{aligned}
0 = \quad & \left[2(\eta - 3)x_{13\kappa} - \Big\{ \frac{\partial}{\partial X_3^\kappa} + (x_{13}^2 + \theta_{13}^2 \bar{\theta}_{13}^2) I^{-1\mu}{}_\kappa(z_{13}) \Big(\frac{\partial \Theta_3^\alpha}{\partial x_{13}^\mu} \frac{\partial^-}{\partial \Theta_3^\alpha} + \frac{\partial \overline{\Theta}_3^{\dot{\alpha}}}{\partial x_{13}^\mu} \frac{\partial^+}{\partial \overline{\Theta}_3^{\dot{\alpha}}} \Big) \Big\} \right] \\
& \times \frac{\mathcal{H}^{\kappa\nu\lambda}(Z_3)}{(X_3^2 + \Theta_3^2 \overline{\Theta}_3^2)^{\eta/2}}
\end{aligned}
\tag{B.17}
$$

When $\theta_i = \bar{\theta}_i = 0,\ i = 1,2,3$, this equation becomes

$$2(3-\eta)x_{13\kappa}\frac{\mathcal{H}^{\kappa\nu\lambda}(Z_3)}{(X_3^2)^{\eta/3}} = \frac{\partial}{\partial X_3^\kappa}\frac{\mathcal{H}^{\kappa\nu\lambda}(Z_3)}{(X_3^2)^{\eta/3}} \tag{B.18}$$

Left hand side is a function of x_{13}, X_3 and right hand side is a function of X_3. Hence for this equation to hold for any value of $x_{13}, X_3,\ \eta$ should be 3. Hence eq.(B.17) becomes

$$0 = \left\{\frac{\partial}{\partial X_3^\kappa} + (x_{13}^2 + \theta_{13}^2\bar{\theta}_{13}^2)I^{-1\mu}{}_\kappa(z_{13})\Big(\frac{\partial\Theta_3^\alpha}{\partial x_{13}^\mu}\frac{\partial^-}{\partial\Theta_3^\alpha} + \frac{\partial\overline{\Theta}_3^{\dot\alpha}}{\partial x_{13}^\mu}\frac{\partial^+}{\partial\overline{\Theta}_3^{\dot\alpha}}\Big)\right\}\frac{\mathcal{H}^{\kappa\nu\lambda}(Z_3)}{(X_3^2 + \Theta_3^2\overline{\Theta}_3^2)^{3/2}} \tag{B.19}$$

$(x_{13}^2 + \theta_{13}^2\bar{\theta}_{13}^2)I^{-1\mu}{}_\kappa(z_{13})\frac{\partial\Theta_3^\alpha}{\partial x_{13}^\mu}$ is a function of $x_{13}, \bar{\theta}_{13}, \theta_{13}\bar{\theta}^2$ and $(x_{13}^2 + \theta_{13}^2\bar{\theta}_{13}^2)I^{-1\mu}{}_\kappa(z_{13})\frac{\partial\overline{\Theta}_3^{\dot\alpha}}{\partial x_{13}^\mu}$ is a function of $x_{13}, \theta_{13}, \bar{\theta}_{13}\theta_{13}^2$. Thus each of the three terms appearing in eq.(B.19) should be zero and so

$$0 = \partial_\kappa\frac{\mathcal{H}^{\kappa\nu\lambda}(z)}{(x^2+\theta^2\bar{\theta}^2)^{3/2}} \tag{B.20}$$

$$0 = \tilde{\sigma}_\kappa^{\dot\alpha\alpha}\partial_\alpha^-\frac{\mathcal{H}^{\kappa\nu\lambda}(z)}{(x^2+\theta^2\bar{\theta}^2)^{3/2}} \tag{B.21}$$

$$0 = \tilde{\sigma}_\kappa^{\dot\alpha\alpha}\bar{\partial}_{\dot\alpha}^+\frac{\mathcal{H}^{\kappa\nu\lambda}(z)}{(x^2+\theta^2\bar{\theta}^2)^{3/2}} \tag{B.22}$$

Nevertheless eq.(B.22) is just the "strong" conservation equation. The other two equations can be obtained from eq.(B.22) by taking its complex conjugate and using $\{\partial_\alpha^-, \bar{\partial}_{\dot\alpha}^+\} = 2i\partial_{\alpha\dot\alpha}$.

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